

On the computation of the HNF of a module over the ring of integers of a number field

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Abstract

We present a variation of the modular algorithm for computing the Hermite normal form of an \mathcal{O}_K -module presented by Cohen [7], where \mathcal{O}_K is the ring of integers of a number field K . An approach presented in [7] based on reductions modulo ideals was conjectured to run in polynomial time by Cohen, but so far, no such proof was available in the literature. In this paper, we present a modification of the approach of [7] to prevent the coefficient swell and we rigorously assess its complexity with respect to the size of the input and the invariants of the field K .

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1. Introduction

Algorithms for modules over the rational integers such as the Hermite normal form algorithm are at the core of all methods for computations with rings

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and ideals in finite extensions of the rational numbers. Following the growing interest in relative extensions, that is, finite extensions of number fields, the structure of modules over Dedekind domains became important. On the theoretical side, it was well known that the framework of finitely generated projective modules was well suited for these problems, but explicit algorithms were lacking for a long time. Based on the pioneering work of Bosma and Pohst [4], the computation of a Hermite normal form (HNF) over principal ideal domains was generalized to finitely generated modules over Dedekind domains by Cohen [7] (for a comparison between the work of Bosma–Pohst and Cohen, see [12, Chap. 6]). It was conjectured that Cohen’s algorithm [7, Algorithm 3.2] for computing this so-called pseudo-Hermite normal form (pseudo-HNF) has polynomial complexity (see [7, Remark after Algorithm 3.2]): “...and it seems plausible that ...this algorithm is, in fact, polynomial-time.” The polynomial complexity of a (modified) version of Cohen’s algorithm was conjectured in the folklore but not formally proved until the preliminary version of this study in the ISSAC proceedings [3]. The difficulties in establishing this formally were two-fold: The original algorithm does not control the size of the coefficient ideals, and, most of the underlying field and ideal operations themselves have not been analyzed completely. While the ideal operations, which are based on Hermite normal forms over the rational integers, are known to have polynomial complexity, the exact complexity was previously not investigated in detail hence a byproduct of this discussion is a computational model for algebraic number fields together with an analysis of basic field and ideal operations.

Based on our careful analysis we also compare the complexity of algorithms for finitely generated projective modules over the ring of integers \mathcal{O}_K of a number field K based on the structure as \mathcal{O}_K -modules with algorithms based on the structure as free \mathbb{Z} -modules of larger rank. In practice, algebraic number fields L of large degree are carefully constructed as relative extensions $\mathbb{Q} \subseteq K \subseteq L$. The computational complexity of element and ideal operations in L depend on both $d = [K : \mathbb{Q}]$ and $n = [L : K]$. Ideals of the ring of integers \mathcal{O}_L of L are naturally \mathbb{Z} -modules of rank dn and therefore ideal arithmetic is reduced to computation of \mathbb{Z} -modules of rank dn . On the other hand, the ring of integers \mathcal{O}_L and its ideals are finitely generated projective modules of rank n over the Dedekind domain \mathcal{O}_K . Thus the ideal arithmetic in \mathcal{O}_L can be performed using the pseudo-HNF algorithm and it is only natural then to ask which method to prefer.

In addition, Fieker and Stehlé’s recent algorithm for computing a reduced basis of \mathcal{O}_K -modules relies on the conjectured possibility to compute a pseudo-HNF for an \mathcal{O}_K -module with polynomial complexity [10, Th. 1]. This allows a reduction algorithm for \mathcal{O}_K -modules with polynomial complexity, similar to the LLL algorithm for \mathbb{Z} -modules.

In the same way as for \mathbb{Z} -modules, where the HNF can be used to compute the Smith normal form, the pseudo-HNF enables us to determine a pseudo-Smith normal form. The pseudo-Smith normal form gives the structure of torsion \mathcal{O}_K -modules, and is used to study the quotient of two modules. Applications include the investigation of Galois cohomology [15].

In all of our algorithms and the analysis we assume that the maximal order, \mathcal{O}_K , is part of the input.

Our contribution

Let K be a number field with ring of integers \mathcal{O}_K . We present in this paper the first algorithm for computing a pseudo-HNF of an \mathcal{O}_K -module which has a proven polynomial complexity. Our algorithm is based on the modular approach of Cohen [8, Chap. 1] extending and correcting the version from the ISSAC proceedings [3]. We derive bounds on its complexity with respect to the size of the input, the rank of the module and the invariants of the field.

As every \mathcal{O}_K -module is naturally a \mathbb{Z} -module (of larger rank), we then compare the complexity of module operations as \mathcal{O}_K -modules to the complexity of the same operations as \mathbb{Z} -modules. In particular, we show that the complexity of the \mathcal{O}_K -module approach with respect to the degree of the field K is (much) worse than in the \mathbb{Z} -module approach. This is due to the (bad) performance of our key tool: An algorithm to establish tight bounds on the norms of the coefficient ideals during the pseudo-HNF algorithm.

As an application of our algorithm, we extend the techniques to also give an algorithm with polynomial complexity to compute the pseudo-Smith normal form associated to \mathcal{O}_K -modules, which is a constructive variant of the elementary divisor theorem for modules over \mathcal{O}_K . Similarly to the pseudo-HNF, this is the first algorithm for this task that is proven to have polynomial complexity.

Outline

In order to discuss the complexity of our algorithms, we start by introducing our computational model and natural representations of the involved objects. Next, suitable definitions for size of the objects are introduced and the behavior under necessary operations is analyzed.

Once the size of the objects is settled, we proceed to develop algorithms for all basic operations we will encounter and prove complexity results for all algorithms. In particular, this section contains algorithms and their complexity for most common ideal operations in number fields. While most of the methods are folklore, this is the first time their complexity has been stated.

Next, the key new technique, the normalization of the coefficient ideals, is introduced and analyzed. Finally, after all the tools are in place, we move to the module theory. Similar to other modular algorithms, we first need to find a suitable modulus. Here this is the determinantal ideal, which is the product of fractional ideals and the determinant of a matrix with entries in \mathcal{O}_K . In Section 5 we present a Chinese remainder theorem based algorithm for the determinant computation over rings of integers and analyze its complexity.

In Section 6, we get to the main result: An explicit algorithm that will compute a pseudo-HNF for any full rank module over the ring of integers. The module is specified via a pseudo-generating system (pairs of fractional ideals of the number field K and vectors in K^m). Under the assumption that the module has full rank and that it is contained in \mathcal{O}_K^m , we prove the following (see Theorem 34):

Theorem. *There exists an algorithm (Algorithm 5), that given n pseudo-generators of an \mathcal{O}_K -module of full rank contained in \mathcal{O}_K^m , computes a pseudo-HNF with polynomial complexity.*

Actually, a more precise version is proven. The exact dependency on the ring of integers \mathcal{O}_K , the dimension of the module and the size of the generators is presented. Note that we assume that certain data of the number field K is precomputed, including an integral basis of the ring of integers (see Section 3).

In the final section, we apply the pseudo-HNF algorithm to derive a pseudo-Smith normal form algorithm and analyze its complexity, achieving polynomial time complexity as well (Algorithm 6 and Proposition 43).

2. Preliminaries

Number fields

Let K be a number field of degree d and signature (r_1, r_2) . That is K admits r_1 real embeddings and $2r_2$ complex embeddings. We can embed K in $K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ and extend all embeddings to $K_{\mathbb{R}}$. The d -dimensional real vector space $K_{\mathbb{R}}$ carries a Hermitian form T_2 defined by $T_2(\alpha, \beta) = \sum_{\sigma} \sigma(\alpha) \overline{\sigma(\beta)}$ for $\alpha, \beta \in K_{\mathbb{R}}$, where the sum runs over all embeddings, and an associated norm $\|\cdot\|$ defined by $\|\alpha\| = \sqrt{T_2(\alpha, \alpha)}$ for $\alpha \in K_{\mathbb{R}}$. The ring \mathcal{O}_K of algebraic integers is the maximal order of K and therefore a \mathbb{Z} -lattice of rank d with $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Q} = K$. Given any \mathbb{Z} -basis $\omega_1, \dots, \omega_d$ of \mathcal{O}_K , the discriminant Δ_K of the number field K is defined as $\Delta_K = \det(\text{Tr}(\omega_i \omega_j)_{i,j})$, where Tr denotes the trace of the finite field extension $\mathbb{Q} \subseteq K$. The norm of an element $\alpha \in K$ is defined by $N(\alpha) = N_{\mathbb{Q}}^K(\alpha) = \prod_{\sigma} \sigma(\alpha)$ and is equal to the usual field norm of the algebraic extension $K \supseteq \mathbb{Q}$. For $\alpha \in K$, M_{α} denotes the $d \times d$ rational matrix corresponding to $K \rightarrow K, \beta \mapsto \alpha\beta$, with respect to a \mathbb{Q} -basis of K and is called the regular representation of α . Here, using a fixed \mathbb{Q} -basis of K , elements are identified with row-vectors in $\mathbb{Q}^{1 \times d}$.

To represent \mathcal{O}_K -modules we rely on a generalization of the notion of ideal, namely the fractional ideals of \mathcal{O}_K . They are defined as finitely generated \mathbb{Z} -submodules of K . When a fractional ideal is contained in \mathcal{O}_K , we refer to it as an integral ideal, which is in fact an ideal of the ring \mathcal{O}_K . Otherwise, for every fractional ideal \mathfrak{a} of \mathcal{O}_K , there exists $r \in \mathbb{Z}_{>0}$ such that $r\mathfrak{a}$ is integral. The minimal positive integer with this property is defined as the denominator of the fractional ideal \mathfrak{a} and is denoted by $\text{den}(\mathfrak{a})$. The sum of two fractional ideals of \mathcal{O}_K is the usual sum as \mathbb{Z} -modules and the product of two fractional ideals $\mathfrak{a}, \mathfrak{b}$ is given by the \mathbb{Z} -module generated by $\alpha\beta$ with $\alpha \in \mathfrak{a}$ and $\beta \in \mathfrak{b}$. The set of fractional ideals of \mathcal{O}_K forms a monoid with identity \mathcal{O}_K and where the inverse of \mathfrak{a} is $\mathfrak{a}^{-1} := \{\alpha \in K \mid \alpha\mathfrak{a} \subseteq \mathcal{O}_K\}$. Each fractional ideal \mathfrak{a} of K is a free \mathbb{Z} -module of rank d and given any \mathbb{Z} -basis matrix $N_{\mathfrak{a}} \in \mathbb{Q}^{d \times d}$ we define the norm $N(\mathfrak{a})$ of \mathfrak{a} to be $|\det(N_{\mathfrak{a}})| \in \mathbb{Q}$. The norm is multiplicative, and in the case \mathfrak{a} is an integral ideal the norm of \mathfrak{a} is equal to $[\mathcal{O}_K : \mathfrak{a}]$, the index of \mathfrak{a} in \mathcal{O}_K . Also note that the absolute value of the norm of $\alpha \in K$ agrees with the norm of the principal ideal $\alpha\mathcal{O}_K$.

\mathcal{O}_K -modules and the pseudo-Hermite normal form over Dedekind domains

In order to describe the structure of modules over Dedekind domains we rely on the notion of pseudoness introduced by Cohen [7], see also [8, Chapter 1]. Note that, different to [7], our modules are generated by row vectors instead of column vectors and we therefore perform row operations. Let M be a non-zero finitely generated torsion-free \mathcal{O}_K -module and $V = K \otimes_{\mathcal{O}_K} M$, a finite dimensional K -vector space containing M . An indexed family $(\alpha_i, \mathfrak{a}_i)_{1 \leq i \leq n}$ consisting of $\alpha_i \in V$ and fractional ideals \mathfrak{a}_i of K is called a pseudo-generating system of M if

$$M = \mathfrak{a}_1 \alpha_1 + \cdots + \mathfrak{a}_n \alpha_n$$

and a pseudo-basis of M if

$$M = \mathfrak{a}_1 \alpha_1 \oplus \cdots \oplus \mathfrak{a}_n \alpha_n.$$

A pair (A, I) consisting of a matrix $A \in K^{n \times m}$ and a list of fractional ideals $I = (\mathfrak{a}_i)_{1 \leq i \leq n}$ is called a pseudo-matrix. Denoting by $A_1, \dots, A_n \in K^{1 \times m}$ the n rows of A , the sum $\sum_{i=1}^n \mathfrak{a}_i A_i$ is a finitely generated torsion-free \mathcal{O}_K -module associated to this pseudo-matrix. Conversely every finitely generated torsion-free module M gives rise to a pseudo-matrix whose associated module is M . In case of finitely generated torsion-free modules over principal ideal domains, the task of finding a basis of the module can be reduced to finding the Hermite normal form (HNF) of the associated matrix. If the base ring is a Dedekind domain there exists a canonical form for pseudo-matrices, the pseudo-Hermite normal form (pseudo-HNF), which plays the same role as the HNF for principal ideal domains allowing us to construct pseudo-bases from pseudo-generating systems. More precisely let $A \in K^{n \times m}$ be of rank m , (A, I) a pseudo-matrix and M the associated \mathcal{O}_K -module. Then there exists an $n \times n$ matrix $U = (u_{i,j})_{i,j}$ over K and n non-zero fractional ideals $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ of K satisfying

1. for all $1 \leq i, j \leq n$ we have $u_{i,j} \in \mathfrak{b}_i^{-1} \mathfrak{a}_j$,
2. the ideals satisfy $\prod_i \mathfrak{a}_i = \det(U) \prod_i \mathfrak{b}_i$,
3. the matrix UA is of the form

$$\begin{pmatrix} H \\ \mathbf{0} \end{pmatrix},$$

where H is an $m \times m$ lower triangular matrix over K with 1's on the diagonal and $\mathbf{0}$ denotes the zero matrix of suitable dimensions.

4. $M = \mathfrak{b}_1 H_1 \oplus \cdots \oplus \mathfrak{b}_n H_n$ where H_1, \dots, H_n are the rows of H .

The pseudo-matrix $(H, (\mathfrak{b}_i)_{1 \leq i \leq n})$ is called a pseudo-Hermite normal form (pseudo-HNF) of (A, I) resp. of M . Note that with this definition, a pseudo-HNF of an \mathcal{O}_K -module is not unique. In [4, 7, 12], reductions of the coefficients of A modulo certain ideals provide uniqueness of the pseudo-HNF when the reduction algorithm is fixed.

Throughout the paper will make the following restriction: We assume that the associated module M is a subset of \mathcal{O}_K^m . For if $M \subseteq K^m$ there exists an integer $k \in \mathbb{Z}_{>0}$ such that $kM \subseteq \mathcal{O}_K^m$. In case of a square pseudo-matrix (A, I) the determinantal ideal $\mathfrak{d}((A, I))$ is defined as to be $\det(A) \prod_{\mathfrak{a} \in I} \mathfrak{a}$. For a pseudo-matrix (A, I) , $A \in \mathcal{O}_K^{n \times m}$ of rank m , we define the determinantal ideal $\mathfrak{d}((A, I))$ to be the gcd of all determinantal ideals of all $m \times m$ sub-pseudo-matrices of (A, I) (see [7]).

3. Size and costs in algebraic number fields

In order to state the complexity of the pseudo-HNF algorithm, we will now describe representations and algorithms of elements and ideals in number fields, which are the objects we have to compute with. The algorithms and representations chosen here are by no means optimal for all problems involving algebraic number fields. We have chosen the linear algebra heavy approach since it allows for efficient algorithms of the normalization of ideals and reduction of elements with respect to ideals, which are crucial steps in the pseudo-HNF algorithm. For different approaches to element arithmetic we refer the interested reader to [6, 4.2] and [1]. For ideal arithmetic (in particular ideal multiplication) fast Las Vegas type algorithm are available making use of a 2-element ideal representation (see [6, 1]). As our aim is a *deterministic* polynomial time pseudo-HNF algorithm, we will not make use of them.

A notion of size.

To ensure that our algorithm for computing a pseudo-HNF basis of an \mathcal{O}_K -module runs in polynomial time, we need a notion of size that bounds the bit size required to represent ideals and field elements. We assume that the maximal order \mathcal{O}_K of K is given by a fixed \mathbb{Z} -basis $\Omega = (\omega_1, \dots, \omega_d)$ with $\omega_1 = 1$.

Size of ideals

A non-zero integral ideal $\mathfrak{a} \subseteq \mathcal{O}_K$ is a d -dimensional \mathbb{Z} -submodule of \mathcal{O}_K and will be represented by its unique (lower triangular) HNF basis $M_{\mathfrak{a}} \in \mathbb{Z}^{d \times d}$ with respect to the fixed integral basis Ω . The size required to store the matrix is therefore bounded by $d^2 \log(|M_{\mathfrak{a}}|)$, where \log denotes the binary logarithm. Since we assume that ω_1 is set to 1 the value $|M_{\mathfrak{a}}|$ is actually equal to $\min\{a \in \mathbb{Z}_{>0} \mid a \in \mathfrak{a}\}$. (For $A = (a_{i,j})_{i,j} \in \mathbb{Z}^{d \times d}$ we denote $\max_{i,j} |a_{i,j}|$ by $|A|$.) The latter is the well known *minimum* of the integral ideal \mathfrak{a} , which is denoted by $\min(\mathfrak{a})$ and can be characterized as the unique positive integer with $\mathfrak{a} \cap \mathbb{Z} = (\min(\mathfrak{a}))$. Based on this observation we define

$$S(\mathfrak{a}) = d^2 \log(\min(\mathfrak{a}))$$

to be the *size* of \mathfrak{a} . If $\mathfrak{a} = \tilde{\mathfrak{a}}/k$ is a fractional ideal of K , where $\mathfrak{a} \subseteq \mathcal{O}_K$ and $k \in \mathbb{Z}_{>0}$ is the denominator of \mathfrak{a} , we define the *size* of \mathfrak{a} by

$$S(\mathfrak{a}) = S(\tilde{\mathfrak{a}}) + d^2 \log(k).$$

The weight d^2 on the denominator is introduced to have a nice behavior with respect to the common ideal operations. Before we show that, we need to recall some basic facts about the minimum of integral ideals. The weight can also be seen as viewing the ideal as given by a rational matrix directly.

Proposition 1. *Let $\mathfrak{a}, \mathfrak{b}$ be integral ideals and $k \in \mathbb{Z}$, $k \neq 0$. Then the following holds:*

1. $\min(\mathfrak{a} + \mathfrak{b})$ divides $\text{GCD}(\min(\mathfrak{a}), \min(\mathfrak{b}))$.
2. $\min(\mathfrak{a}\mathfrak{b})$ divides $\min(\mathfrak{a})\min(\mathfrak{b})$.
3. The denominator of \mathfrak{a}^{-1} is equal to $\min(\mathfrak{a})$.
4. $\min(k\mathfrak{a}) = |k|\min(\mathfrak{a})$.
5. $\min(\mathfrak{a})$ divides $N(\mathfrak{a})$.

Proof. Follows from the definition. □

The properties of the minimum translate easily into corresponding properties of the size of integral ideals. The next proposition shows that in fact the same relations hold also for fractional ideals.

Proposition 2. *Let $\mathfrak{a}, \mathfrak{b} \subseteq K$ be fractional ideals and $m \in \mathbb{Z}$, $m \neq 0$. Then the following holds:*

1. $S(m\mathfrak{a}) \leq S(\mathfrak{a}) + d^2 \log(|m|)$.
2. $S(\mathfrak{a} + \mathfrak{b}) \leq 2(S(\mathfrak{a}) + S(\mathfrak{b}))$.
3. $S(\mathfrak{a}\mathfrak{b}) \leq S(\mathfrak{a}) + S(\mathfrak{b})$
4. $S(\mathfrak{a}^{-1}) \leq 2S(\mathfrak{a})$.

Proof. Note that if \mathfrak{a} and \mathfrak{b} are integral ideals then (1), (2) and (3) follow immediately from the properties of the minimum obtained in Proposition 1. Write $\mathfrak{a} = \tilde{\mathfrak{a}}/k$ and $\mathfrak{b} = \tilde{\mathfrak{b}}/l$ with k and l the denominator of \mathfrak{a} and \mathfrak{b} respectively. (1): We have

$$S(m\mathfrak{a}) \leq S(m\tilde{\mathfrak{a}}) + d^2 \log(k) \leq S(\tilde{\mathfrak{a}}) + d^2 \log(|m|) + d^2 \log(k) = S(\mathfrak{a}) + d^2 \log(|m|).$$

(2): As the sum $\mathfrak{a} + \mathfrak{b}$ is equal to $(l\tilde{\mathfrak{a}} + k\tilde{\mathfrak{b}})/kl$ we obtain

$$\begin{aligned} S(\mathfrak{a} + \mathfrak{b}) &\leq S(l\tilde{\mathfrak{a}} + k\tilde{\mathfrak{b}}) + d^2 \log(kl) \leq S(\tilde{\mathfrak{a}}) + d^2 \log(l) + S(\tilde{\mathfrak{b}}) + d^2 \log(k) + S(\mathfrak{a}) + S(\mathfrak{b}) \\ &= 2(S(\mathfrak{a}) + S(\mathfrak{b})). \end{aligned}$$

(3): We have

$$S(\mathfrak{a}\mathfrak{b}) \leq S(\tilde{\mathfrak{a}}\tilde{\mathfrak{b}}) + d^2 \log(k) + d^2 \log(l) \leq S(\mathfrak{a}) + S(\mathfrak{b}).$$

(4): Consider first the integral case: We know that $\min(\tilde{\mathfrak{a}}) \in \tilde{\mathfrak{a}}$. Thus the principal ideal $(\min(\tilde{\mathfrak{a}}))$ is divided by $\tilde{\mathfrak{a}}$ and there exists an integral ideal \mathfrak{b} with $(\min(\tilde{\mathfrak{a}})) = \tilde{\mathfrak{a}}\mathfrak{b}$, i. e.,

$$\tilde{\mathfrak{a}}^{-1} = \frac{\mathfrak{b}}{\min(\tilde{\mathfrak{a}})}.$$

Note that $\min(\tilde{\mathfrak{a}}) \in \mathfrak{b}$ and therefore $\min(\mathfrak{b}) \leq \min(\tilde{\mathfrak{a}})$. As $\min(\tilde{\mathfrak{a}})$ is the denominator of $\tilde{\mathfrak{a}}^{-1}$ by Proposition 1 (4) we obtain

$$S(\tilde{\mathfrak{a}}^{-1}) = S(\mathfrak{b}) + d^2 \log(\min(\tilde{\mathfrak{a}})) \leq 2S(\tilde{\mathfrak{a}}).$$

Returning to the general case we have $\mathfrak{a}^{-1} = k\tilde{\mathfrak{a}}^{-1}$. Then

$$S(\mathfrak{a}^{-1}) = S(\tilde{\mathfrak{a}}^{-1}) + d^2 \log(k) \leq 2S(\tilde{\mathfrak{a}}) + 2d^2 \log(k) = 2S(\mathfrak{a}).$$

□

Size of elements.

The integral basis Ω allows us to represent an integral element $\alpha \in \mathcal{O}_K$ by its coefficient vector $(a_1, \dots, a_d) \in \mathbb{Z}^d$ satisfying $\alpha = \sum_{i=1}^d a_i \alpha_i$. The size to store the element α is therefore bounded by

$$S(\alpha) = d \max_i \log(|a_i|),$$

which we call the *size* of α with respect to Ω . This can be faithfully generalized to elements $\alpha \in K$. Writing $\alpha = \tilde{\alpha}/k$ with $k \in \mathbb{Z}_{>0}$ the denominator of α we define

$$S(\alpha) = S(\tilde{\alpha}) + d \log(k)$$

to be the size of α . Similarly to the ideals above, as added the weight d to the denominator to achieve a nicer transformation behavior under the standard operations. Its justification also comes from viewing elements in K as rational vectors rather than integral elements with a common denominator.

In order to relate our function S to the multiplicative structure on K we need to recall that the notion of size of elements is closely related to norms on the \mathbb{R} -vector space $K_{\mathbb{R}}$. More precisely, the fixed integral basis Ω gives rise to an isomorphism

$$\Phi: K_{\mathbb{R}} \rightarrow \mathbb{R}^d, \sum_{i=1}^d a_i \omega_i \mapsto (a_1, \dots, a_d),$$

onto the d -dimensional real vector space. Equipping \mathbb{R}^d with the ∞ -norm we have $d \log(\|\Phi(\alpha)\|_{\infty}) = S(\alpha)$ for $\alpha \in \mathcal{O}_K$. But this is not the only way to identify $K_{\mathbb{R}}$ with a normed real vector space. Denote the r_1 real embeddings by $(\sigma_i)_{1 \leq i \leq r_1}$ and the $2r_2$ complex embeddings by $(\sigma_i)_{r_1+1 \leq i \leq r_1+2r_2}$. We use the usual ordering of the complex embeddings, such that $\sigma_{r_1+k} = \overline{\sigma}_k$ for $r_1 < k \leq r_1 + r_2$. Using these embeddings we define

$$\begin{aligned} \Psi: K_{\mathbb{R}} &\longrightarrow \mathbb{R}^{r_1} \times \mathbb{R}^{2r_2} \\ \alpha &\longmapsto (\sigma_i(\alpha))_{1 \leq i \leq r_1}, (\operatorname{Re} \sigma_i(\alpha) + \operatorname{Im} \sigma_i(\alpha), \operatorname{Re} \sigma_i(\alpha) - \operatorname{Im} \sigma_i(\alpha))_{r_1 < i \leq r_1+2r_2} \end{aligned}$$

yielding $\|\Psi(\alpha)\|_2 = \|\alpha\|$ for $\alpha \in K$, where $\|\cdot\|_2$ denotes the 2-norm on $\mathbb{R}^{r_1+2r_2}$. Since \mathbb{R} is complete, any two norms on $K_{\mathbb{R}}$ are equivalent. Thus there exists constants $C_1, C_2 \in \mathbb{R}_{>0}$ depending on K and the chosen basis Ω with

$$\frac{1}{C_2} \|\Phi(\alpha)\|_{\infty} \leq \|\alpha\| \leq C_1 \|\Phi(\alpha)\|_{\infty}, \quad (1)$$

for all $\alpha \in K$. Moreover we have the inequalities

$$\|\alpha\| \leq \sqrt{d} \max_{\sigma} |\sigma(\alpha)|, \quad \max_{\sigma} |\sigma(\alpha)| \leq \|\alpha\|, \quad (2)$$

for all $\alpha \in K$ and applying the geometric arithmetic mean inequality yields

$$|N(\alpha)| \leq \frac{\|\alpha\|^d}{d^{d/2}}. \quad (3)$$

Another important characteristic of an integral basis Ω is the size of the structure constants $(m_{i,j}^k)_{i,j,k}$, which are defined by the relations

$$\omega_i \omega_j = \sum_{k=1}^d m_{i,j}^k \omega_k$$

for $1 \leq i, j \leq d$. We denote the maximum value $\max_{i,j,k} |m_{i,j}^k|$ by C_3 .

Remark 3. *Note that there is a situation in which we are able to estimate the constants C_1, C_2, C_3 . Assume that Ω is LLL-reduced with respect to T_2 and LLL parameter c . Then by [1, Proposition 5.1] the basis Ω satisfies*

$$\|\omega_i\|^2 \leq \left(d^{-(i-1)} c^{d(d-1)/2} |\Delta_K| \right)^{1/(d-i+1)}$$

for all $1 \leq i \leq d$. Moreover the structure constants satisfy

$$|m_{i,j}^k| \leq \frac{c^{3d(d-1)/4}}{d^{d-(1/2)}} |\Delta_K| \quad 1 \leq i, j, k \leq d,$$

and thus we can choose

$$C_1 = \max_i \left(d^{-(i-1)} c^{d(d-1)/2} |\Delta_K| \right)^{1/2(d-i+1)}, \quad C_3 = \frac{c^{3d(d-1)/4}}{d^{d-(1/2)}}.$$

By [10, Lemma 2] we have $\|\Phi(\alpha)\|_{\infty} \leq 2^{3d/2} \|\alpha\|$ for all $\alpha \in K$ allowing for $C_2 = 2^{3d/2}$.

Using the preceding discussion we can now describe the relation between size and the multiplicative structure of \mathcal{O}_K . If $\alpha = \sum_{i=1}^d a_i \omega_i$ and $\beta = \sum_{j=1}^d b_j \omega_j$ are integral elements the product $\alpha\beta$ is equal to $\sum_{k=1}^d c_k \omega_k$ with

$$c_k = \sum_{i=1}^d a_i \sum_{j=1}^d b_j m_{i,j}^k.$$

Thus for the size of $\alpha\beta$ we obtain

$$S(\alpha\beta) \leq S(\alpha) + S(\beta) + 2d \log(d) + d \log(C_3).$$

The constant $2d \log(d) + d \log(C_3)$ therefore measures the increase of size when multiplying two integral elements.

The second multiplicative operation is the inversion of integral elements. Let $\alpha^{-1} = \beta/k$ with $k \in \mathbb{Z}_{>0}$ the denominator of α^{-1} and $\beta \in \mathcal{O}_K$. Using $k \leq |\mathbf{N}(\alpha)|$ and Inequality (3) we obtain $\log(k) \leq d \log(C_1) + d \log(\|\alpha\|_\infty) - \frac{d}{2} \log(d)$. Since

$$|\sigma(\beta)| = \frac{\sigma(k)}{|\sigma(\alpha)|} = \frac{k}{|\sigma(\alpha)|} \leq \frac{|\mathbf{N}(\alpha)|}{|\sigma(\alpha)|} = \prod_{\tau \neq \sigma} |\tau(\alpha)| \leq \|\alpha\|^{d-1}$$

for every embedding σ we get $\|\beta\| \leq \sqrt{d} \|\alpha\|^{d-1}$ by Inequality (2). Combining this with the estimate for the denominator yields

$$\mathbf{S}(\alpha^{-1}) = d \log(k) + \mathbf{S}(\beta) \leq d \mathbf{S}(\alpha) + d^2 \log(C_1) + d \log(C_2).$$

Again we see that there is a constant depending on Ω describing the increase of size during element inversion. We define C_Ω by

$$C_\Omega = \max\{2d \log(d) + d \log(C_3), d^2 \log(C_1) + d \log(C_2)\}$$

to obtain a constant incorporating both operations. Since we work with a fixed basis we drop the Ω from the index and denote this constant just by C . So far the obtained bounds on the size are only valid for integral elements and it remains to prove similar relations for the whole of K . We begin with the multiplicative structure.

Proposition 4. *For all $\alpha, \beta \in K$ and $m \in \mathbb{Z}$, $m \neq 0$, the following holds:*

1. $\mathbf{S}(m\alpha) = \mathbf{S}(\alpha) + d \log(|m|)$.
2. $\mathbf{S}(\alpha\beta) \leq \mathbf{S}(\alpha) + \mathbf{S}(\beta) + C$,
3. $\mathbf{S}(\alpha^{-1}) \leq d \mathbf{S}(\alpha) + C$.

Proof. We write $\alpha = \tilde{\alpha}/k$ and $\beta = \tilde{\beta}/l$ with k and l the denominator of α and β respectively. Note that by the choice of C items (2) and (3) hold for integral elements. (1): From the definition of the size it follows that $\mathbf{S}(m\tilde{\alpha}) = \mathbf{S}(\tilde{\alpha}) + d \log(|m|)$. Since the denominator of $m\alpha$ is bounded by k we have

$$\mathbf{S}(m\alpha) \leq \mathbf{S}(m\tilde{\alpha}) + d \log(k) = \mathbf{S}(\alpha) + d \log(|m|).$$

(2): Since the denominator of $\alpha\beta$ is bounded by kl we obtain

$$\mathbf{S}(\alpha\beta) \leq \mathbf{S}(\tilde{\alpha}\tilde{\beta}) + d \log(kl) \leq \mathbf{S}(\alpha) + \mathbf{S}(\beta) + C.$$

(3): The inverse of α is equal to $k\tilde{\alpha}^{-1}$. Therefore using (1) we get

$$\mathbf{S}(\alpha^{-1}) = \mathbf{S}(\tilde{\alpha}^{-1}) + d \log(k) = d \log(k) + \mathbf{S}(\tilde{\alpha}) + C \leq d \mathbf{S}(\alpha) + C.$$

□

We now investigate the additive structure.

Proposition 5. *If α and β are elements of K then $\mathbf{S}(\alpha + \beta) \leq 2(\mathbf{S}(\alpha) + \mathbf{S}(\beta))$.*

Proof. It is easy to see that $S(\alpha + \beta) \leq S(\alpha) + S(\beta)$ if α and β are integral elements. Now write $\alpha = \tilde{\alpha}/k$ and $\beta = \tilde{\beta}/l$ with k and l the denominator of α and β respectively. Then we obtain $S(l\tilde{\alpha} + k\tilde{\beta}) \leq S(\tilde{\alpha}) + S(\tilde{\beta}) + d\log(k) + d\log(l) = S(\alpha) + S(\beta)$ and finally

$$S(\alpha + \beta) \leq S(l\tilde{\alpha} + k\tilde{\beta}) + d\log(kl) \leq 2(S(\alpha) + S(\beta)).$$

□

Finally we need the mixed operation between ideals and elements.

Proposition 6. *Let $\alpha \in K$ and $\mathfrak{a} \subseteq K$ be a fractional ideal. Then $S(\alpha\mathfrak{a}) \leq S(\mathfrak{a}) + d^2S(\alpha) + dC$.*

Proof. We consider first the integral case $\alpha \in \mathcal{O}_K$ and $\mathfrak{a} \subseteq \mathcal{O}_K$. Using Inequalities (1) and (3) the minimum of the principal ideal (α) can be bounded by $C_1^d \|\alpha\|_\infty^d$. Thus we have

$$S(\alpha\mathfrak{a}) = d^2 \log(\min(\alpha\mathfrak{a})) \leq d^2 \log(\min(\alpha)) + d^2 \log(\min(\mathfrak{a})) \leq d^2S(\alpha) + S(\mathfrak{a}) + dC.$$

Now let $\alpha = \tilde{\alpha}/k$ and $\mathfrak{a} = \tilde{\mathfrak{a}}/l$ with k and l the denominator of α and \mathfrak{a} respectively. Using the integral case we obtain

$$\begin{aligned} S(\alpha\mathfrak{a}) &\leq S(\tilde{\alpha}\tilde{\mathfrak{a}}) + d^2 \log(kl) \leq S(\tilde{\mathfrak{a}}) + d^2 \log(l) + d^2S(\tilde{\alpha}) + d^2 \log(k) + dC \\ &= S(\mathfrak{a}) + d^2S(\alpha) + dC. \end{aligned}$$

□

Calculating in K .

In this section, we evaluate the complexity of the basic operations performed during the pseudo-HNF algorithm. To simplify the representation of complexity results, we use soft-Oh notation \tilde{O} : We have $f \in \tilde{O}(g)$ if and only if there exists $k \in \mathbb{Z}_{>0}$ such that $f \in O(g(\log(g))^k)$. We multiply two integers of bit size B with complexity in $\tilde{O}(B)$ using the Schönhage–Strassen algorithm [19]. While the addition of such integers is in $O(B)$, their division has complexity in $\tilde{O}(B)$.

As most of our algorithms are going to be based on linear algebra over rings, mainly \mathbb{Z} , we start by collecting the complexity of the used algorithms. The basic problem of determining the unique solution $x \in \mathbb{Q}^n$ to the equation $Ax = b$ with $A \in \mathbb{Z}^{n \times n}$ non-singular, $b \in \mathbb{Z}^n$ can be done using Dixon's p -adic algorithm [9] in $\tilde{O}(n^3(\log(|A|) + \log(|b|)))$.

As we represent integral ideals using the HNF basis, the computation of this form is at the heart of ideal arithmetic. Note, that in contrast to the standard case in the literature [11, 22] we do not want to state the complexity in terms of the determinant (or multiples thereof) but in terms of the elementary divisors. As we will see, in our applications, we always know small multiples of the elementary divisors and thus obtain tighter bounds. Important to the algorithms is the notion of a Howell form of a matrix as defined in [13]. The Howell form generalizes the Hermite normal form to $\mathbb{Z}/\lambda\mathbb{Z}$ and restores uniqueness in the

presence of zero divisors. For a matrix $A \in \mathbb{Z}^{n \times m}$ of rank m we denote by $\text{HNF}(A)$ the unique Hermite form of the matrix (with the off-diagonal elements reduced into the positive residue system modulo the diagonal), while $\text{How}_\lambda(A)$ will denote the Howell form for $A \in (\mathbb{Z}/\lambda\mathbb{Z})^{n \times m}$. In [23] a naive algorithm is given that computes $\text{How}_\lambda(A)$ in time $O(m^2 \max(n, m))$ operations in $\mathbb{Z}/\lambda\mathbb{Z}$. We also need the following facts:

Lemma 7. *Let $A \in \mathbb{Z}^{n \times m}$ and $\lambda \in \mathbb{Z}$ such that $\lambda\mathbb{Z}^m \subseteq [A]_{\mathbb{Z}}$ where $[A]_{\mathbb{Z}}$ denotes the \mathbb{Z} -module generated by the rows of A . Then the following holds:*

1. *We have*

$$\text{HNF}(A) = \text{HNF}\left(\frac{A}{\lambda I_m}\right).$$

2. *We have $\text{HNF}(A) = \text{How}_{\lambda^2}(A)$, that is, the canonical lifting of the Howell form over $\mathbb{Z}/\lambda^2\mathbb{Z}$ yields the Hermite form over \mathbb{Z} .*

Proof. Since, by assumption,

$$[A]_{\mathbb{Z}} = \left[\frac{A}{\lambda I_m} \right]_{\mathbb{Z}}$$

and the Hermite form is an invariant of the module, the first claim is clear.

To show the second claim, it is sufficient to show that the reduction of $\text{HNF}(A)$ modulo λ^2 has all the properties of the Howell form. Once this is clear, the claim follows from the uniqueness of the Howell form as an invariant of the $\mathbb{Z}/\lambda^2\mathbb{Z}$ module and the fact that all entries in $\text{HNF}(A)$ are non-negative and bounded by λ . The only property of the Howell form that needs verification, is the last claim: any vector in $[A]_{\mathbb{Z}/\lambda^2\mathbb{Z}}$ having first coefficients zero is in the span of the last rows of the Howell form. This follows directly from the Hermite form: any lift of such a vector is a sum of a vector in $\lambda^2\mathbb{Z}$ and an element in $[A]$ starting with the same number of zeroes as the initial element. Such an element is clearly in the span of the last rows of $\text{HNF}(A)$ since the Hermite form describes a basis and the linear combination carries over modulo λ^2 . The other properties of the Howell form are immediate: the reduction modulo the diagonal as well as the overall shape is directly inherited from the Hermite form. The final property, the normalization of the diagonal namely to divide λ^2 follows too from the Hermite form: since $\lambda\mathbb{Z}^m$ is contained in the module, the diagonal entries of the Hermite form have to be divisors of λ , hence of λ^2 . We note, that the reason we chose λ^2 over λ is to avoid problems with vanishing diagonal elements: as all diagonal entries of the Hermite form are divisors of λ , none of them can vanish in $\mathbb{Z}/\lambda^2\mathbb{Z}$. \square

We can now derive the complexity of the HNF computation in terms of λ .

Corollary 8. *Let $A \in \mathbb{Z}^{n \times m}$ be a matrix and $\lambda \in \mathbb{Z}$ such that $\lambda\mathbb{Z}^m \subseteq [A]_{\mathbb{Z}}$. Then the Hermite normal form of A can be computed with complexity in $\tilde{O}(mn \log(|A|) + m^2 \max(m, n) \log(|\lambda|))$.*

Proof. The Lemma 7 links the Hermite normal form to the Howell form, while Storjohann's naive algorithm [23] will compute the Howell form with the complexity as stated. \square

We will see, that in our applications, we naturally know and control a multiple of the largest elementary divisor, hence we can use this rather than the determinant in our complexity analysis.

Note that due to Storjohann and Mulders [23] there exists asymptotically fast algorithms for computing the Howell form based on fast matrix multiplication. Since our pseudo-HNF algorithm is a generalization of a non-asymptotically fast HNF algorithm over the integers and eventually we want to compare our pseudo-HNF algorithm with the absolute HNF algorithm it is only reasonable to not use asymptotically fast algorithms for the underlying element and ideal arithmetic.

Concerning our number field K , we take the following precomputed data for granted:

- An integral basis $\Omega = (\omega_i)_i$ of the maximal order \mathcal{O}_K satisfying $\omega_1 = 1$.
- The structure constants $M = (m_{i,j}^k)_{i,j,k}$ of Ω .
- The matrix DT^{-1} , where $T = (\text{Tr}(\omega_i \omega_j))_{i,j}$ and D is the denominator of T^{-1} . Moreover using [10, Theorem 3] we compute a LLL-reduced 2-element representation (δ_1, δ_2) of the ideal \mathfrak{B} generated by the rows of DT^{-1} with the property

$$\|\delta_i\| \leq 4(2^{\frac{d}{2}})^8 |\Delta_K|^{\frac{2}{d}} (C_1)^4, \quad \text{i.e., } S(\delta_i) \in \tilde{O}(\log(|\Delta_K|) + C)$$

for $i = 1, 2$. In addition we compute the regular representations M_{δ_1} and M_{δ_2} .

- A primitive element of K with minimal polynomial $f = X^d + \sum_{i=0}^{d-1} a_i X^i \in \mathbb{Z}[X]$, such that $\log(\max_i |a_i|) \leq C$ and $\log(|\text{disc}(f)|) \in \tilde{O}(C)$. Such an element can be found as follows: By a theorem of Sonn and Zassenhaus [21] there exist $\varepsilon_1, \dots, \varepsilon_d \in \{0, 1\}$ such that $\alpha = \sum_{i=1}^d \varepsilon_i \omega_i \in \mathcal{O}_K$ is a primitive element of the field extension $\mathbb{Q} \subseteq K$. Note that with the currently known methods finding such an element is exponentially costly with respect to d . Applying the d embeddings σ_j we obtain

$$|\sigma_j(\alpha)| \leq d \max_i |\sigma_j(\omega_i)| \leq d \max_i \|\omega_i\| \leq dC_1.$$

Using these estimates for the conjugates of α we get the following bound on the coefficients of the minimal polynomial $f = X^d + \sum_{i=0}^{d-1} a_i X^i \in \mathbb{Z}[X]$ of α : Since the elements $\sigma_j(\alpha)$, $1 \leq j \leq d$, are exactly the roots of f we obtain

$$|a_i| = |s_i(\sigma_1(\alpha), \dots, \sigma_d(\alpha))| \leq \binom{d}{i} \max_j |\sigma_j(\alpha)|^i \leq d^d \max_j |\sigma_j(\alpha)|^d \leq d^d d^d C_1^d,$$

for $0 \leq i \leq d-1$, where s_i denotes the elementary symmetric polynomial of degree i . Therefore the height $|f| = \max_i |a_i|$ of f can be estimated by

$$\log(|f|) = \max_i \log(|a_i|) \leq 2d \log(d) + d \log(C_1) \leq C,$$

As we have a bound for the absolute values of its roots, we can moreover derive the following estimate for the discriminant of f :

$$|\text{disc}(f)| = \prod_{i < j} |\sigma_i(\alpha) - \sigma_j(\alpha)|^2 \leq |\max_j 2\sigma_j(\alpha)|^{d^2} \leq 2^{d^2} \max_j |\sigma_j(\alpha)|^{d^2}.$$

Taking logarithms on both sides we obtain

$$\log(|\text{disc}(f)|) \in O(\log(d^2 \max_j |\sigma_j(\alpha)|)) \subseteq O(d^2(\log(d) + \log(C_1))) \subseteq \tilde{O}(C).$$

We do not impose any further restrictions on our integral basis Ω . All dependency on Ω is captured by $C = C_\Omega$.

Field arithmetic

During our pseudo-HNF computation we need to perform additions, multiplications, and inversions of elements of K . Although algorithms for these operations are well known (see [6, 1]) and many implementations can be found, there is a lack of references on the complexity. While multiplication in \mathcal{O}_K was investigated by Belabas [1], all the other operations are missing. We address the complexity issues in the rest of this section and begin with the additive structure.

Proposition 9. *Let $\alpha, \beta \in K$ and $m \in \mathbb{Z}$. We can*

1. *compute the product $m\alpha$ with complexity in $\tilde{O}(S(\alpha) + d \log(|m|))$.*
2. *compute the quotient α/m with complexity in $\tilde{O}(S(\alpha) + \log(|m|))$.*
3. *compute the sum $\alpha + \beta$ with complexity in $O(S(\alpha) + S(\beta))$.*

Proof. Let us write $\alpha = \tilde{\alpha}/k$ and $\beta = \tilde{\beta}/l$ with k and l the denominator of α and β respectively.

(1): Computing the GCD g of m and k as well as k/g and m/g have complexity in $\tilde{O}(S(\alpha)/d + \log(m))$. This is followed by computing $(m/g)\tilde{\alpha}$ which has complexity in $\tilde{O}(S(\alpha) + d \log(m))$ and dominates the computation.

(2): Let (a_1, \dots, a_d) be the coefficient vector of $\tilde{\alpha}$ and $g = \text{GCD}(m, a_1, \dots, a_d)$. The quotient α/m is then given by $(\tilde{\alpha}/g)/(k \cdot m/g)$. As the costs of computing g are in $\tilde{O}(\log(|m|) + d \log(\|\tilde{\alpha}\|_\infty))$ and the products can be computed in $\tilde{O}(d \log(\|\tilde{\alpha}\|_\infty))$ and $\tilde{O}(\log(|m|) + \log(k))$ the claim follows.

(3): The complexity obviously holds for integral elements. By (1) the computation of $l\tilde{\alpha}$ and $k\tilde{\beta}$ has complexity in $\tilde{O}(S(\alpha) + S(\beta))$ and the complexity of adding $l\tilde{\alpha}$ and $k\tilde{\beta}$ is in $\tilde{O}(S(\alpha) + S(\beta))$. Computing kl has complexity in $\tilde{O}(S(\alpha)/d + S(\beta)/d)$. The last thing we have to do is making sure that the coefficients of the numerator and the denominator are coprime. This is done by d GCD computations and d divisions with complexity in $\tilde{O}(d(S(\alpha)/d + S(\beta)/d))$. \square

Proposition 10. *Let $\alpha, \beta, \alpha_1, \dots, \alpha_n \in K$, $\gamma \in \mathcal{O}_K$ an integral element and $m \in \mathbb{Z}$. We can*

1. *compute the regular representation M_γ of γ with complexity in $\tilde{O}(d^2\mathbf{S}(\gamma) + d^2C)$.*
2. *compute the product $\alpha\beta$ with complexity in $\tilde{O}(d\mathbf{S}(\alpha) + d\mathbf{S}(\beta) + dC)$ if the regular representation of the numerator of α is known.*
3. *compute the product $\alpha\beta$ with complexity in $\tilde{O}(d^2\mathbf{S}(\alpha) + d\mathbf{S}(\beta) + d^2C)$*
4. *compute the products $\alpha\alpha_i$, $1 \leq i \leq n$, with complexity in $\tilde{O}(d(d+n)\mathbf{S}(\alpha) + dn \max_i \mathbf{S}(\alpha_i) + d(d+n)C)$.*
5. *compute the inverse α^{-1} with complexity in $\tilde{O}(d^2\mathbf{S}(\alpha) + d^2C)$ if $\alpha \neq 0$.*

Proof. Let us write $\alpha = \tilde{\alpha}/k$ and $\beta = \tilde{\beta}/l$ with k and l the denominator of α and β respectively.

(1): If $(c_1, \dots, c_d) \in \mathbb{Z}^d$ denotes the coefficient vector of γ , the regular representation is given by

$$M_\gamma = \left(\sum_{j=1}^d c_j m_{ij}^k \right)_{i,k}.$$

Thus computing M_γ involves d^3 multiplications (and additions) and the overall complexity is in $\tilde{O}(d^2\mathbf{S}(\gamma) + d^3 \log(C_3)) = \tilde{O}(d^2\mathbf{S}(\gamma) + d^2C)$.

(2): Let (a_1, \dots, a_d) and (b_1, \dots, b_d) be the coefficient vectors of $\tilde{\alpha}$ and $\tilde{\beta}$ respectively. The coefficients of the product $\tilde{\alpha}\tilde{\beta} = \sum_{i=1}^d c_i \omega_i$ are given by

$$(c_1, \dots, c_d) = (b_1, \dots, b_d)M_{\tilde{\alpha}}.$$

Hence the product is obtained by d^2 multiplications. As the matrix $M_{\tilde{\alpha}}$ satisfies $\log(|M_{\tilde{\alpha}}|) \in \tilde{O}(\mathbf{S}(\tilde{\alpha})/d + C/d)$ this has complexity in $\tilde{O}(d\mathbf{S}(\tilde{\alpha}) + d\mathbf{S}(\tilde{\beta}) + dC)$. Since taking care of denominators is less expensive this step dominates the computation.

(3): Use (1) and (2).

(4): We first evaluate the complexity of inverting the integral element $\tilde{\alpha}$. In this case the coefficients b_1, \dots, b_n of the element $\delta \in K$ with $\tilde{\alpha}\delta = 1$ satisfy

$$(b_1, \dots, b_d)M_{\tilde{\alpha}} = (1, 0, \dots, 0).$$

Thus inverting $\tilde{\alpha}$ boils down to calculating the regular representation of $\tilde{\alpha}$ and finding the unique rational solution of a linear system of d integer equations. By (1) the computation of $M_{\tilde{\alpha}}$ has complexity in $\tilde{O}(d^2\mathbf{S}(\tilde{\alpha}) + d^2C)$ and the entries of $M_{\tilde{\alpha}}$ satisfy $\log(|M_{\tilde{\alpha}}|) \in O(\mathbf{S}(\tilde{\alpha})/d + C/d)$. Using Dixon's algorithm solving the system then has complexity in $\tilde{O}(d^2\mathbf{S}(\alpha) + d^2C)$. Now the inverse of α is given by $\alpha^{-1} = k\tilde{\alpha}^{-1}$. Since $\mathbf{S}(\tilde{\alpha}^{-1}) \leq d\mathbf{S}(\alpha) + C$ the complexity to compute $k\tilde{\alpha}^{-1}$ is in $\tilde{O}(d\mathbf{S}(\alpha) + C)$. \square

Ideal arithmetic

By definition integral ideals are represented by their unique HNF with respect to the fixed integral basis. Therefore operations with ideals are mainly HNF computations which are accelerated by the availability of a multiple of the corresponding largest elementary divisor. More precisely let \mathfrak{a} be an integral ideal with HNF $A \in \mathbb{Z}^{d \times d}$. As $\min(\mathfrak{a})$ is an element of \mathfrak{a} we know that $\min(\mathfrak{a})\omega_i \in \mathfrak{a}$ for all $1 \leq i \leq d$. On the side of the \mathbb{Z} -module structure this implies $\min(\mathfrak{a})\mathbb{Z}^d \subseteq [A]_{\mathbb{Z}}$ allowing us to work modulo $\min(\mathfrak{a})^2$ during the HNF computation by Lemma 7. The following lemma for computing the sum of ideals illustrates these ideas.

Lemma 11. *Let \mathfrak{a} and \mathfrak{b} be fractional ideals and $m \in \mathbb{Z}$. We can*

1. *compute $m\mathfrak{a}$ with complexity in $\tilde{O}(S(\mathfrak{a}) + d^2 \log(|m|))$.*
2. *compute the sum $\mathfrak{a} + \mathfrak{b}$ with complexity in $\tilde{O}(d(S(\mathfrak{a}) + S(\mathfrak{b})))$.*

Proof. We write $\mathfrak{a} = \tilde{\mathfrak{a}}/k$ and $\mathfrak{b} = \tilde{\mathfrak{b}}/l$ with k and l the denominator of \mathfrak{a} and \mathfrak{b} respectively.

(1): We first have to compute the GCD g of m and k . Together with the division of k and m by g this has complexity in $\tilde{O}(\log(|m|) + \log(k))$. Finally we have to multiply the HNF matrix of $\tilde{\mathfrak{a}}$ with m/g taking d^2 multiplications with integers of size bounded by $S(\tilde{\mathfrak{a}})/d^2 + \log(|m|)$. In total we obtain a complexity in $\tilde{O}(S(\mathfrak{a}) + d^2 \log(|m|))$.

(2): We first consider the case of integral ideals $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{b}}$. The HNF basis of $\tilde{\mathfrak{a}} + \tilde{\mathfrak{b}}$ is obtained by computing the HNF of the concatenation $(M_{\tilde{\mathfrak{a}}}^t | M_{\tilde{\mathfrak{b}}}^t)^t$. As the minimum of $\tilde{\mathfrak{a}} + \tilde{\mathfrak{b}}$ divides $\text{GCD}(\min(\tilde{\mathfrak{a}}), \min(\tilde{\mathfrak{b}}))$ by Corollary 8 this computation can be done with complexity in

$$\begin{aligned} & \tilde{O}((2d)d(\log(\min(\tilde{\mathfrak{a}})) + \log(\min(\tilde{\mathfrak{b}}))) + (2d)d^2 \log(\text{GCD}(\min(\tilde{\mathfrak{a}}), \min(\tilde{\mathfrak{b}})))) \\ & \subseteq \tilde{O}(d \min(S(\tilde{\mathfrak{a}}), S(\tilde{\mathfrak{b}}))). \end{aligned}$$

Now consider the fractional case. By (1) and the integral case computing $l\tilde{\mathfrak{a}} + k\tilde{\mathfrak{b}}$ has complexity in $\tilde{O}(d(S(\mathfrak{a}) + S(\mathfrak{b})))$. Since this dominates the denominator computation we obtain an overall complexity as claimed. \square

Proposition 12. *Let $\alpha \in K$ and $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_K$ be integral ideals. We can*

1. *compute $\mathfrak{a}\mathfrak{b}$ with complexity in $\tilde{O}(d^2 S(\mathfrak{a}) + d^2 S(\mathfrak{b}) + d^3 C)$.*
2. *compute $\alpha\mathfrak{a}$ with complexity in $\tilde{O}(d^3 S(\alpha) + dS(\mathfrak{a}) + d^2 C)$.*

Proof. We write $\mathfrak{a} = \tilde{\mathfrak{a}}/k$, $\mathfrak{b} = \tilde{\mathfrak{b}}/l$ and $\alpha = \tilde{\alpha}/m$ with k, l and m the denominator of $\mathfrak{a}, \mathfrak{b}$ and α respectively.

(1): As $\mathfrak{a}\mathfrak{b} = \tilde{\mathfrak{a}}\tilde{\mathfrak{b}}/(kl)$ we first evaluate the complexity of computing $\tilde{\mathfrak{a}}\tilde{\mathfrak{b}}$. Denoting by $(\alpha_i)_i$ and $(\beta_j)_j$ the HNF bases of $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{b}}$ respectively we know that $S(\alpha_i) \leq S(\tilde{\mathfrak{a}})/d$ and $S(\beta_j) \leq S(\tilde{\mathfrak{b}})/d$ respectively. The d^2 elements $(\alpha_i \beta_j)_{i,j}$ form a \mathbb{Z} -generating system of $\tilde{\mathfrak{a}}\tilde{\mathfrak{b}}$ and their computation has complexity in

$$\tilde{O}(d^3(S(\tilde{\mathfrak{a}})/d + S(\tilde{\mathfrak{b}})/d) + d^3 C) = \tilde{O}(d^2 S(\tilde{\mathfrak{a}}) + d^2 S(\tilde{\mathfrak{b}}) + d^3 C).$$

The matrix M of this generating system then satisfies $\log(|M|) \leq S(\tilde{\mathbf{a}})/d^2 + S(\tilde{\mathbf{b}})/d^2 + C/d$. As the minimum of $\tilde{\mathbf{a}}\tilde{\mathbf{b}}$ divides $\min(\tilde{\mathbf{a}})\min(\tilde{\mathbf{b}})$ the final HNF computation has complexity in

$$\tilde{O}(d^2S(\tilde{\mathbf{a}}) + d^2S(\tilde{\mathbf{b}}) + d^2C).$$

As denominator computation is dominated by these steps the claim holds.

(2): If we denote the HNF basis of $\tilde{\mathbf{a}}$ by $(\alpha_i)_i$ we know that $(\tilde{\alpha}\alpha_i)_i$ forms a \mathbb{Z} -generating system of the ideal $\tilde{\alpha}\tilde{\mathbf{a}}$. Computing the d products $\tilde{\alpha}\alpha_i$ for $1 \leq i \leq d$ has complexity in $\tilde{O}(d^2S(\tilde{\alpha}) + dS(\tilde{\mathbf{a}}) + d^2C)$ since we have to compute the regular representation of $\tilde{\alpha}$ only once. If M denotes the matrix corresponding to this generating system of $\tilde{\alpha}\tilde{\mathbf{a}}$ we know that $\log(|M|) \leq S(\tilde{\alpha})/d + S(\tilde{\mathbf{a}})/d^2 + C/d$. Before computing the HNF matrix, we take care of the denominator. Computing kl , the GCD of kl and the entries of the matrix M and dividing kl and M by the GCD has complexity in $\tilde{O}(dS(\alpha) + S(\mathbf{a}) + dC)$. As we know the regular representation of $\tilde{\alpha}$ we also know the minimum of the principal ideal (α) . In particular we know $\min((\tilde{\alpha}))\min(\tilde{\mathbf{a}})$ which is a multiple of $\min(\tilde{\alpha}\tilde{\mathbf{a}})$. Using the estimate $S((\tilde{\alpha})) \leq d^2S(\tilde{\alpha}) + dC$ (see proof of Proposition 6) and Corollary 8 the final HNF can be computed with complexity in

$$\tilde{O}(dS(\mathbf{a}) + d^3S(\alpha) + d^2C).$$

□

Finally we need to invert ideals. We use a slightly modified version of [1, Algorithm 5.3] (which itself is a modified version of [6, Algorithm 4.8.21]), exploiting the fact that

$$\mathfrak{a}^{-1} = \{\alpha \in K \mid \text{Tr}(\alpha\mathfrak{D}^{-1}\mathfrak{a}) \subseteq \mathbb{Z}\},$$

where \mathfrak{D} denotes the different of K . Recall that \mathfrak{D}^{-1} is a fractional ideal with (fractional) basis matrix $T^{-1} \in \mathbb{Q}^{d \times d}$, where $T = (\text{Tr}(\omega_i\omega_j))_{i,j}$. In order to evaluate the complexity of ideal inversion we need a bound on the size of mT^{-1} where m denotes the denominator of T^{-1} , that is, $m = \min(\mathfrak{D})$. Since by Cramer's rule we know that $|mT^{-1}| \leq d^d|T|^d$ it remains to consider $|T|$. By definition the trace of an element $\alpha \in K$ is given by the trace of its regular representation, $\text{Tr}(\alpha) = \text{Tr}(M_\alpha)$. In case of a basis element $\alpha = \omega_k$ for some $1 \leq k \leq d$ the entries of M_α are just structure constants m_{ij}^k and therefore $|\text{Tr}(\omega_k)| \leq dC_3$. Applying this to $\text{Tr}(\omega_i\omega_j)$ for $1 \leq i, j \leq d$ yields

$$|\text{Tr}(\omega_i\omega_j)| \leq \sum_{k=1}^d |m_{ij}^k| |\text{Tr}(\omega_k)| \leq d^2C_3^2$$

and therefore

$$\log(|mT^{-1}|) \leq 2d\log(d) + 2d\log(C_3) \in O(C).$$

In addition note that $\min(\mathfrak{D})$ divides the norm of \mathfrak{D} , which is just $|\Delta_K|$.

Proposition 13. *Let \mathfrak{a} be a fractional ideal. Then we can compute \mathfrak{a}^{-1} with complexity in $\tilde{O}(dS(\mathfrak{a}) + d^3 \log(|\Delta_K|) + d^2 C)$.*

Proof. We use the same notation as in the preceding discussion. Let us first consider the integral case $\mathfrak{a} \subseteq \mathcal{O}_K$. Recall that the denominator of \mathfrak{a}^{-1} is just $\min(\mathfrak{a})$ and need not be computed. Denote by $(\alpha_i)_i$ the HNF basis of \mathfrak{a} and by \mathfrak{B} the integral ideal $m\mathfrak{D}^{-1}$. We first have to compute $\mathfrak{a}\mathfrak{B}$. Using the precomputed 2-element representation $\mathfrak{B} = (\delta_1, \delta_2)$ this amounts to compute $2d$ products $\alpha_i \delta_j$, $1 \leq i \leq d$, $1 \leq j \leq 2$. As we have also precomputed the regular representation of δ_1 and δ_2 this has complexity in $\tilde{O}(dS(\mathfrak{a}) + d^2 \log(|\Delta_K|) + d^2 C)$ and yields a matrix $M \in \mathbb{Z}^{2d \times d}$ with $\log(|M|) \leq S(\mathfrak{a})/d^2 + \log(|\Delta_K|)/d + C/d$. The cost of computing the HNF H of M is therefore in $\tilde{O}(dS(\mathfrak{a}) + d^3 \log(|\Delta_K|) + dC)$, where we use that the minimum of $\mathfrak{a}\mathfrak{B}$ divides the $\min(\mathfrak{a})|\Delta_K|$. A transposed basis matrix of the numerator of \mathfrak{a}^{-1} is then obtained as the solution $X \in \mathbb{Z}^{d \times d}$ of the equation $HX = \min(\mathfrak{a})(mT^{-1})$. Note that the triangular shape of H allows us to recover X by back substitution. Since $\min(\mathfrak{a})^2$ is contained in the span of X we can work modulo $\min(\mathfrak{a})^2$. The estimates $\log(|H|) \leq \log(\min(\mathfrak{a})|\Delta_K|)$ and $\log(mT^{-1}) \in O(C)$ show that the initial reduction has complexity in $\tilde{O}(S(\mathfrak{a}) + d^2 \log(|\Delta_K|) + d^2 C)$. For each column of X the back substitution itself then has a complexity in $\tilde{O}(d^2 \min(\mathfrak{a}))$ yielding a complexity of $\tilde{O}(dS(\mathfrak{a}))$ in total for obtaining X . Finally we need to compute the HNF of X^t which has complexity in $\tilde{O}(d^2 \log(|X|) + d^3 \log(\min(\mathfrak{a}))) \subseteq \tilde{O}(dS(\mathfrak{a}))$.

Now let $\mathfrak{a} = \tilde{\mathfrak{a}}/k$ be a fractional ideal with denominator k . As $S(\tilde{\mathfrak{a}}^{-1}) \leq 2S(\tilde{\mathfrak{a}})$ the computation of $k\tilde{\mathfrak{a}}^{-1}$ has complexity in $\tilde{O}(S(\tilde{\mathfrak{a}}) + d^2 \log(k)) = \tilde{O}(S(\mathfrak{a}))$ and the claim follows. \square

4. Normalization and reduction

There are different strategies for dealing with coefficient explosion during classical Hermite normal form algorithms over \mathbb{Z} . One strategy, which is used by Hafner and McCurley [11] exploits the fact that the whole computation can be done modulo some multiple of the determinant of the associated lattice (in case of a square non-singular matrix this is just the determinant of the matrix). Fortunately the same holds for the pseudo-HNF over Dedekind domains and therefore we are allowed to use reduction modulo (different) integral ideals involving the determinantal ideal. Unfortunately these ideals are in general not generated by a single rational integer, making the notion of reduction more difficult. We will use the approach of Cohen [7, Algorithm 2.12] with a different reduction algorithm and provide a rigorous complexity analysis. The reduction is accompanied by a normalization algorithm, which bounds the size of the coefficient ideals and heavily depends on lattice reduction. The output of both algorithms, reduction and normalization, depends on the size of the lattice reduced basis and the smaller the lattice basis the smaller the output. There are various lattice reduction algorithms and in general the smaller the basis the worse the complexity of the algorithm. Thus one has to balance between smallness and efficiency. Instead of the L^2 algorithm of Nguyen and Stehlé [16],

which has complexity quadratic in the size of the input, we rely on the nearly linear \tilde{L}^1 -algorithm of Novocin, Stehlé and Villard which provides a lattice basis satisfying a weakened LLL condition. More precisely, for $\Xi = (\delta, \eta, \delta)$ with $\eta \in [\frac{1}{2}, 1)$, $\theta \geq 0$ and $\delta \in (\eta^2, 1]$, the notion of an Ξ -LLL reduced basis is defined in [5]. Setting $\ell = (\theta\eta + \sqrt{(1 + \theta^2)\delta - \eta^2})(\delta - \eta^2)^{-1}$ it is proved in [5, Theorem 5.4] that an Ξ -LLL reduced basis (b_1, \dots, b_n) of a lattice L of rank n in an Euclidean space satisfies

$$\begin{aligned} \|b_1\| &\leq \ell^{n-1} \lambda(L), \\ \|b_1\| &\leq \ell^{(n-1)/2} |\det(L)|^{1/n}, \\ \prod_{1 \leq j \leq n} \|b_j\| &\leq \ell^{n(n-1)/2} |\det(L)|, \end{aligned} \tag{4}$$

where $\det(L)$ resp. $\lambda(L)$ denotes the determinant resp. the first minimum of the lattice L . Using this weakened LLL condition Novocin, Stehlé and Villard ([17]) construct an algorithm, \tilde{L}^1 , with the following property ([17, Theorem 7]): Given a matrix $B \in \mathbb{Z}^{d \times d}$ with rows b_1, \dots, b_j satisfying $\max_j \|b_j\| \leq 2^\beta$, the \tilde{L}^1 algorithm returns a Ξ -LLL reduced basis of the lattice associated to B within $\tilde{O}(d^5 \beta)$ operations.

Rounded lattice reduction

Since the \tilde{L}^1 algorithm operates only on integral input, we now describe how to use this algorithm in our case, where the input is a lattice with real basis. We closely follow the ideas and arguments of [1, Section 4], where a similar analysis was done for LLL reduction. Let $G = (T_2(\omega_i, \omega_j))_{1 \leq i, j \leq d} \in \mathbb{R}^{d \times d}$ be the Gram matrix of T_2 with respect to the integral basis of K . Let RR^t be the Cholesky decomposition of G and $e \in \mathbb{Z}_{\geq 1}$ an integer such that the integral matrix $R^{(e)} = \lceil 2^e R \rceil \in \mathbb{Z}^{d \times d}$ has full rank. Thus for an element $\alpha = \sum_{i=1}^d a_i \omega_i$ of K with coefficient vector $X = (a_1, \dots, a_d)$ we have $\|\alpha\| = \|XR\|_2$. We now set $T_2^{(e)}(\alpha) = \|XR^{(e)}\|$. Then $T_2^{(e)}$ is an integral approximation of $2^{2e} T_2$ with integral Gram matrix. While in general the application of lattice reduction with respect to this approximated form $T_2^{(e)}$ does not yield a reduced basis with respect to T_2 , the basis one obtains satisfies size estimates similar to (4).

Proposition 14. *Let L be a sublattice of \mathcal{O}_K and let $(\alpha_i)_{1 \leq i \leq d}$ be a Ξ -LLL reduced basis for L with respect to $T_2^{(e)}$. Then*

$$\begin{aligned} \|\alpha_1\| &\leq C_{\Omega, d, e} \cdot \ell^{(d-1)/2} \cdot |\det(L, T_2)|^{1/d} \text{ and} \\ \prod_{1 \leq i \leq n} \|\alpha_i\| &\leq C_{\Omega, d, e}^d \cdot \ell^{d(d-1)/2} \cdot |\det(L, T_2)|, \end{aligned}$$

with a constant $C_{\Omega, d, e}$ depending on the integral basis Ω , the field degree d , e , and not depending on L , and which satisfies $C_{\Omega, d, e} \rightarrow 1$ for $e \rightarrow \infty$. Here with $\det(L, T_2)$ we denote the determinant of L with respect to T_2 .

Proof. The proof is similar to the proof of [1, Proposition 4.2]. We set $S = (R^{(e)})^{-1}$ and write $R^{(e)} = 2^e R + \varepsilon$ with $\varepsilon \in \mathbb{R}^{d \times d}$. Let X_i be the coefficient vector of α_i and let $Y_i = X_i R^{(e)}$. Since the (α_i) are a Ξ -LLL reduced basis of L with respect to $T_2^{(e)}$ we have

$$\prod_{1 \leq i \leq d} \|Y_i\|_2 = \prod_{1 \leq i \leq d} \sqrt{T_2^{(e)}(\alpha_i)} \leq \ell^{d(d-1)/2} |\det(L, T_2^{(e)})|.$$

As in [1] we have $2^e \|\alpha_i\| \leq (1 + \|\varepsilon S\|_2) \|Y_i\|$. Using the fact that

$$\det(L, T_2^{(e)}) = \det(L, T_2) \frac{\det(R^{(e)})}{\det(R)}$$

we obtain

$$\prod_{1 \leq i \leq d} \|\alpha_i\| \leq (1 + \|\varepsilon S\|)^d \frac{1}{2^{de}} \prod_{1 \leq i \leq d} \|Y_i\|_2 \leq (1 + \|\varepsilon S\|_2)^d \left(\frac{\det(R^{(e)})}{2^{ed} \det(R)} \right) \ell^{d(d-1)/2} |\det(L, T_2)|.$$

Now the claim follows from setting

$$C_{\Omega, d, e} = (1 + \|\varepsilon S\|_2) \left(\frac{\det(R^{(e)})}{2^{ed} \det(R)} \right)^{1/d}$$

and [1, Corollary 4.3]. \square

We call the basis (α_i) as in the statement an approximated Ξ -LLL reduced basis of L . To use this in our setting, we add the computation of $\lceil 2^e R \rceil \in \mathbb{Z}^{d \times d}$ to the list of precomputed data. We treat $C_{\Omega, d, e}$ as well as $\lceil 2^e R \rceil$ as constants during the complexity analysis. Moreover to simplify the exposition we replace ℓ by $C_{\Omega, d, e}^{2/(d-1)} \ell$, so that the last two equations of (4) hold for an approximated Ξ -LLL reduced basis. To compute such a basis for an integral ideal \mathfrak{a} we just have to apply the \tilde{L}^1 algorithm to the matrix $\lceil 2^e R \rceil M_{\mathfrak{a}}$, which has a complexity in $\tilde{O}(d^5 \log(\min(\mathfrak{a})))$.

Reduction with respect to fractional ideals.

Using the approximated reduced lattices, we now describe how to use this to reduce elements modulo ideals. We begin with the integral case and assume that \mathfrak{a} is an integral ideal and $\alpha \in \mathcal{O}$. The goal of the reduction algorithm is to replace the element α by $\bar{\alpha} \in K$ such that $\alpha - \bar{\alpha} \in \mathfrak{a}$ and $\bar{\alpha}$ is small with respect to $N(\mathfrak{a})$ and T_2 -norm. Let $(\alpha_i)_i$ be a \mathbb{Z} -basis of \mathfrak{a} and $\alpha = \sum_i a_i \alpha_i$ the representation of α in the \mathbb{Q} -basis $(\alpha_i)_i$ of K . The element $\bar{\alpha}$ defined by $\bar{\alpha} = \sum_i (a_i - \lceil a_i \rceil) \alpha_i$ satisfies

$$\alpha - \bar{\alpha} = \sum_{i=1}^d \lceil a_i \rceil \alpha_i \in \mathfrak{a} \quad \text{and} \quad \|\bar{\alpha}\| \leq \sum_i |a_i - \lceil a_i \rceil| \|\alpha_i\| \leq \frac{1}{2} \sum_i \|\alpha_i\| \leq \frac{d}{2} \max_i \|\alpha_i\|.$$

Here, as usual, $\lceil a_i \rceil := \lceil 1 + 1/2 \rceil$ denotes rounding. By the arithmetic-geometric mean inequality we have

$$\|\alpha_j\| \geq \sqrt{d} N(\alpha_j)^{\frac{1}{d}} \geq \sqrt{d} N(\mathfrak{a})^{\frac{1}{d}}$$

for all $1 \leq j \leq d$ and assuming that $(\alpha_i)_i$ is Ξ -LLL reduced, we obtain by (4)

$$\prod_{1 \leq i \leq d} \|\alpha_i\| \leq \ell^{\frac{d(d-1)}{2}} \det(L_{\mathfrak{a}})$$

where $L_{\mathfrak{a}}$ denotes the lattice associated to \mathfrak{a} and $\det(L_{\mathfrak{a}})$ its determinant. Using both inequalities we obtain

$$d^{\frac{d-1}{2}} N(\mathfrak{a})^{\frac{d-1}{d}} \|\alpha_j\| \leq \prod_{1 \leq i \leq d} \|\alpha_i\| \leq \ell^{\frac{d(d-1)}{2}} \det(L_{\mathfrak{a}})$$

and thus

$$\|\alpha_j\| \leq \ell^{\frac{d(d-1)}{2}} d^{-\frac{d-1}{2}} N(\mathfrak{a})^{-\frac{d-1}{d}} \det(L_{\mathfrak{a}}) \leq \sqrt{d} \ell^{\frac{d(d-1)}{2}} N(\mathfrak{a})^{\frac{1}{d}} \sqrt{|\Delta_K|} \quad (5)$$

for all $1 \leq j \leq d$. Hence we are able to bound $\|\bar{\alpha}\|$ in terms of $N(\mathfrak{a})$.

Consider now the fractional case with $\alpha = \beta/k$ and $\mathfrak{a} = \mathfrak{b}/l$. Then the above consideration applied to $l\beta$ and $k\mathfrak{b}$ yields an element $\bar{\alpha}$ with

$$\alpha - \bar{\alpha}/(kl) \in \mathfrak{a}$$

and

$$\|\bar{\alpha}/(kl)\| \leq d^{3/2} \ell^{d(d-1)/2} N(\mathfrak{a})^{1/d} \sqrt{|\Delta_K|}$$

To compute $\bar{\alpha}/(kl)$ we proceed as follows. Denote by $A = (a_1, \dots, a_n)$ the coefficient vector of β with respect to the integral basis. As $l\beta \subseteq \mathfrak{b}$ there exists $Y \in \mathbb{Z}^d$ such that $LY = A$, that is, Y is the coefficient vector of $l\beta$ with respect to the basis matrix $L \in \mathbb{Z}^{d \times d}$ of \mathfrak{b} . Dividing by k we obtain Y/k , which is then the coefficient vector of $l\beta$ with respect to the basis matrix kL of $k\mathfrak{b}$. Finally the coefficient vector of $\bar{\alpha}/(kl)$ is given by

$$\frac{1}{kl}(kL) \left(\frac{Y}{k} - \left\lceil \frac{Y}{k} \right\rceil \right) = \frac{1}{l} L \left(\frac{Y \bmod k}{k} \right) = \frac{1}{kl} L(Y \bmod k).$$

This procedure is summarized in Algorithm 1.

Proposition 15. *Algorithm 1 is correct and has complexity in*

$$\tilde{O}(d^3 S(\mathfrak{a}) + d^2 S(\alpha) + d^3 \log(|\Delta_K|) + d^3 C).$$

The size of the output $\tilde{\alpha}$ satisfies

$$\|\tilde{\alpha}\| \leq d^{\frac{3}{2}} \ell^{\frac{d(d-1)}{2}} N(\mathfrak{a})^{\frac{1}{d}} \sqrt{|\Delta_K|}.$$

Moreover if the approximated reduced basis of the numerator of \mathfrak{a} is known, then the reduction of α has complexity in

$$\tilde{O}(dS(\mathfrak{a}) + d^2 S(\alpha) + d^2 C + d^3 \log(|\Delta_K|)).$$

Algorithm 1 Reduction modulo integral ideals

Input: $\alpha \in K$, fractional ideal \mathfrak{a} of K .

Output: $\tilde{\alpha} \in K$ such that $\alpha - \tilde{\alpha} \in \mathfrak{a}$ and $\|\tilde{\alpha}\| \leq d^{3/2} \ell^{d(d-1)/2} N(\mathfrak{a})^{1/d} \sqrt{|\Delta_K|}$.

- 1: Let $\alpha = \beta/k$ and $\mathfrak{a} = \mathfrak{b}/l$.
 - 2: Compute an approximated Ξ -LLL reduced basis matrix $L \in \mathbb{Z}^{d \times d}$ of \mathfrak{b} using the \tilde{L}^1 -algorithm.
 - 3: Solve $LY = lA$ for $Y \in \mathbb{Z}^n$, where A is the coefficient vector of β .
 - 4: Compute $Y \bmod k$ and $Z = 1/(kl)L(Y \bmod k)$.
 - 5: **return** The element corresponding to Z .
-

Proof. As correctness was already shown we just have to do the cost analysis. The \tilde{L}^1 -algorithm allows us to compute L with complexity in $\tilde{O}(d^5 \log(\min(\mathfrak{a})))$. Write $B_L = \log(|L|)$ and $B_\beta = \log(\|\beta\|_\infty)$. Applying Dixon's algorithm to compute Y has costs in $\tilde{O}(d^3(B_L + B_\beta + \log(l)))$ and invoking Cramer's rule we see that $|Y| \leq d^d B_L^d B_\beta \log(l)$, that is, $\log(|Y|) \in \tilde{O}(dB_L + B_\beta + \log(l))$. Therefore the d divisions required to compute $Y \bmod k$ have complexity in $\tilde{O}(dB_L + B_\beta + \log(l))$. Since $|Y \bmod k| \leq k$ the matrix vector multiplications consists of d^2 multiplications of integers of size bounded by $\tilde{O}(B_L + \log(k))$ and the output satisfies $\log(|L(Y \bmod k)|) \in \tilde{O}(\log(k) + B_L)$. Finally the product kl , as well as d GCDs and divisions with $L(Y \bmod k)$ need to be computed with complexity in $\tilde{O}(d(B_L + \log(k) + \log(l)))$. Without the computation of the approximated reduced basis we have in total a complexity in

$$\tilde{O}(d^3 B_L + d^3 B_\beta + d \log(k) + d \log(l))$$

which simplifies to

$$\tilde{O}(dS(\mathfrak{a}) + d^2 S(\alpha) + d^3 \log(|\Delta_K|) + d^2 C)$$

using the bound $B_L \in \tilde{O}(\min(\mathfrak{b}) + d^2 + \log(C_2) + \log(|\Delta|))$ derived from (5). Since the complexity of the \tilde{L}^1 -algorithm is in $\tilde{O}(d^3 S(\mathfrak{a}) + d^3 C)$ the claim follows. \square

Remark 16.

1. Note that the computation of the approximated reduced basis gives a big contribution to the overall complexity of Algorithm 1. It is therefore important to compute the approximated reduced basis only once, when reducing lots of elements of K modulo a fixed ideal. More precisely the reduction of n elements $\alpha_1, \dots, \alpha_n \in K$ can be done in

$$\tilde{O}(d^3 S(\mathfrak{a}) + ndS(\mathfrak{a}) + nd^2 \max_i S(\alpha_i) + (n + d)d^2 C + nd^3 \log(|\Delta_K|)).$$

2. A reduced element is not necessarily of small size since the T_2 -norm of a field element alone does not control the size of the element. More precisely if α is in K and $k \in \mathbb{Z}_{>0}$ is the denominator of α then we have

$$S(\alpha) = d \log(\|k\alpha\|_\infty) + \log(k) \leq (d + 1) \log(k) + C + \log(\|\alpha\|).$$

Thus in addition we need to control the size of the denominator to ensure that the reduced element is small with respect to S .

Normalization.

The normalization is the key difference between our approach and the one of Cohen [7]. It is the strategy that together with the reduction prevents the coefficient swell by calculating a pseudo-basis for which the ideals are integral with size bounded by invariants of the field. The connection between the size of the integral coefficient ideals and denominators of the matrix entries is seen as follows. Assume that $(A, (\mathfrak{a}_i)_i)$ is a pseudo-matrix of an \mathcal{O}_K -module M and A_i is the i -th row of A . Since we consider only modules M contained in \mathcal{O}_K^n we see that $\mathfrak{a}_i A_i \subseteq \mathcal{O}_K^n$ allowing us to bound the denominators of the entries of A_i by $\min(\mathfrak{a}_i)$.

Since $\mathfrak{a}_i A_i = \alpha \mathfrak{a}_i (1/\alpha) A_i$ we can adjust our coefficient ideals by scalars from K (while multiplying the row with the inverse). Therefore the task is to find an integral ideal \mathfrak{b} such that $\alpha \mathfrak{b}^{-1}$ is principal and $N(\mathfrak{b})$ is small. Basically we just have to find a small integral representative of the ideal class of \mathfrak{a} . The usual proof of the finiteness of the class number provides us with such a small representative and a norm bound involving Minkowski's constant. As this is not suited for algorithmic purposes we handle this problem using Ξ -LLL reduced bases.

We write $\mathfrak{a} = \mathfrak{b}/k$ and $\mathfrak{b}^{-1} = \mathfrak{c}/l$ with k and l the denominator of \mathfrak{a} and \mathfrak{b}^{-1} respectively. Applying the \tilde{L}^1 -algorithm we find an element $\alpha \in \mathfrak{c}$ satisfying

$$\|\alpha\| \leq \ell^{(d-1)/2} |\Delta_K|^{1/(2d)} N(\mathfrak{c})^{1/d}, \quad (6)$$

that is

$$N(\alpha) \leq \ell^{d^2} \sqrt{|\Delta_K|} N(\mathfrak{c}).$$

Then the ideal $\tilde{\mathfrak{a}}$ defined by $\tilde{\mathfrak{a}} = (\alpha/l)k\mathfrak{a}$ is integral since $\alpha \in \mathfrak{c} = l\mathfrak{b}^{-1} = l(k\mathfrak{a})^{-1}$. Moreover its norm satisfies

$$N(\tilde{\mathfrak{a}}) = N(\alpha/l) N(k\mathfrak{a}) = \frac{N(\alpha)}{N(\mathfrak{c})} \leq \ell^{d^2} \sqrt{|\Delta_K|}.$$

and is therefore bounded by invariants of the field.

Algorithm 2 Normalization of a one-dimensional module

Input: $A = (\alpha_1, \dots, \alpha_n) \in K^n$, fractional ideal \mathfrak{a} of K with denominator k .

Output: $\tilde{A} \in K^n$, $\tilde{\mathfrak{a}} \subseteq \mathcal{O}_K$ such that $N(\tilde{\mathfrak{a}}) \leq \ell^{d^2} \sqrt{|\Delta_K|}$ and $\mathfrak{a}A = \tilde{\mathfrak{a}}\tilde{A}$.

- 1: Compute $\mathfrak{b}^{-1} = \mathfrak{c}/l$ where \mathfrak{b} is the numerator of \mathfrak{a} .
 - 2: Let α be the first element of an approximated Ξ -LLL reduced basis of \mathfrak{c} .
 - 3: **return** $l/(k\alpha)A$, $(\alpha/l)k\mathfrak{a}$.
-

Proposition 17. *Algorithm 2 is correct and its output satisfies*

$$\begin{aligned} S(\tilde{\alpha}) &\in \tilde{O} \left(S(\mathfrak{a}) + \max_i S(\alpha_i) + d \log(|\Delta_K|) + dC \right), \\ S(\tilde{\mathfrak{a}}) &\in \tilde{O} \left(d^4 + d^2 \log(|\Delta_K|) \right), \end{aligned}$$

where $\tilde{\alpha} \in K$ is a coefficient of \tilde{A} . Its complexity is in

$$\tilde{O}(d(d^2 + n)\mathbf{S}(\mathbf{a}) + dn \max_i(\mathbf{S}(\alpha_i)) + d^2(d + n)(\log(|\Delta_K|) + C)).$$

Proof. The correctness of the algorithm follows from the preceding discussion. Computing the inverse of \mathbf{b} can be done in $\tilde{O}(d\mathbf{S}(\mathbf{b}) + d^3 \log(|\Delta_K|) + d^2C)$. The output satisfies $\mathbf{S}(\mathbf{c}) \leq \mathbf{S}(\mathbf{b})$ as well as $l \leq \min(\mathbf{b})$. The second step invokes the \tilde{L}^1 -algorithm whose complexity is in $\tilde{O}(d^3\mathbf{S}(\mathbf{c}) + d^3C)$ and which computes a small element $\alpha \in \mathbf{c}$ with the property as in (6). Now this bound on the T_2 -norm translates into $\mathbf{S}(\alpha) \in \tilde{O}(\mathbf{S}(\mathbf{b})/d + \log(|\Delta_K|) + C)$. The element α/l can be computed in $\tilde{O}(\mathbf{S}(\alpha) + \log(l))$ and satisfies $\mathbf{S}(\alpha/l) \leq \mathbf{S}(\alpha) + d \log(l)$. Thus computing the new coefficient ideal $(\alpha/l)k\mathbf{a} = (\alpha/l)\mathbf{b}$ costs $\tilde{O}(d^3\mathbf{S}(\alpha/l) + d\mathbf{S}(\mathbf{b}) + d^2C) \subseteq \tilde{O}(d^2\mathbf{S}(\mathbf{a}) + d^3(\log(|\Delta_K|) + C))$.

It remains to consider the multiplication of A by $l/(k\alpha)$. Inverting α and multiplying α^{-1} by l/k has complexity in $\tilde{O}(d^2\mathbf{S}(\alpha) + d^2C + \log(k) + d \log(l))$. Since $\mathbf{S}(l/(k\alpha)) \in \tilde{O}(d\mathbf{S}(\alpha) + d \log(l) + d \log(k) + C)$ the multiplication with A has complexity in

$$\tilde{O}(d(d + n)(d\mathbf{S}(\alpha) + d \log(l) + d \log(k)) + dn \max_i(\mathbf{S}(\alpha_i)) + d(d + n)C),$$

which reduces to $\tilde{O}(d(d + n)\mathbf{S}(\mathbf{a}) + dn \max_i(\mathbf{S}(\alpha_i)) + d^2(d + n)(\log(|\Delta_K|) + C))$. Now the claim follows. \square

5. Computation of determinants over rings of integers

As already mentioned the important step during our pseudo-HNF algorithm is the ability to reduce the entries modulo some integral ideal involving the determinantal ideal of the module. The algorithms presented in this section describe how to obtain the determinantal ideal in case it is not known in advance. We first describe a polynomial algorithm for computing the determinant of a square matrix over \mathcal{O}_K . Already for integer matrices computing determinants is a rather involved task, see [14] for a survey of different approaches and their complexity. Performing very well in practice and being a deterministic polynomial algorithm we present a determinant algorithm for matrices over \mathcal{O}_K which is based on the small primes modular approach.

Bounding the size of the output.

The underlying idea of a modular determinant algorithm is the possibility to bound the size of the result before the actual computation. For a matrix $A = (a_{ij})_{i,j} \in \mathcal{O}_K^{n \times n}$ denote by $|A|$ the bound $\max_{i,j} \{\|a_{ij}\|_\infty\}$.

Lemma 18. *Let $A = (a_{ij})_{i,j} \in \mathcal{O}_K^{n \times n}$. Then $\|\det(A)\|_\infty \leq n^n C_1 C_2^n |A|^n$, that is,*

$$\log(\|\det(A)\|_\infty) \in O(n \log(n|A|) + nC).$$

Proof. We have $\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n a_{i, \sigma(i)}$ and therefore

$$\|\det(A)\|_\infty \leq C_1 \|\det(A)\| \leq C_1 n! \max_{i,j} (\|a_{ij}\|)^n \leq C_1 C_2^n n^n |A|.$$

□

Recall that in the absolute case $\mathcal{O}_K = \mathbb{Z}$ we get the same bound without the term nC and we can immediately formulate a modular algorithm for determinant computations: Find $B \in \mathbb{Z}_{>0}$ such that $|\det(A)| < B/2$. Then compute the determinant of the matrix modulo B and obtain a number $d \leq B/2$ such that $d \equiv \det(A) \pmod{B}$. Since d and $\det(A)$ are bounded by $B/2$ they must be equal.

Let us now show that the recovering technique can be applied to algebraic integers in place of rational integers.

Lemma 19. *Let $\alpha = \sum_{i=1}^d a_i \omega_i$ and $\beta = \sum_{i=1}^d b_i \omega_i$ be two algebraic integers in \mathcal{O}_K . Assume there exists $B \in \mathbb{R}_{>0}$ such that $|a_i|, |b_j| < B/2$ for all $1 \leq i, j \leq d$ and $\alpha \equiv \beta \pmod{(B)}$. Then $\alpha = \beta$.*

Proof. Since $(\omega_i)_i$ is a \mathbb{Z} -basis of \mathcal{O} , the family $(B\omega_i)_i$ is a \mathbb{Z} -basis of the principal ideal (B) . Hence $\alpha \equiv \beta \pmod{(B)}$ is equivalent to the divisibility of $a_i - b_i$ by B for all $1 \leq i \leq d$. Using the coefficient bound we obtain

$$0 \leq |a_i - b_i| \leq |a_i| + |b_i| < B.$$

We conclude that $a_i = b_i$ for all $1 \leq i \leq d$, that is, $\alpha = \beta$. □

We can now proceed as in the integer case. After computing the determinant modulo several primes p we combine the results via the Chinese remainder theorem. As soon as the product exceeds the a priori bound from Lemma 18 we can recover the actual value using Lemma 19. As $\mathcal{O}_K/(p)$ is in general not a nice ring to work with, we want to decompose (p) into prime ideals \mathfrak{p} of \mathcal{O}_K allowing for computations in the finite field $\mathcal{O}_K/\mathfrak{p}$. Again the result modulo (p) can be obtained invoking the Chinese remainder theorem. We address the computational complexity of this two stage Chinese remaindering in the following section.

Chinese remaindering for rational primes and prime ideals over ring of integers.

Let $p \in \mathbb{Z}_{>0}$ be rational prime. By the theory of maximal orders in number fields, see [6, 4.6.2], there exists a factorization

$$(p) = \prod_{i=1}^g \mathfrak{p}_i^{e_i}$$

into pairwise different prime ideals $\mathfrak{p}_i \subseteq \mathcal{O}_K$ with exponents $e_i \in \mathbb{Z}_{>0}$. Moreover there exists $f_i \in \mathbb{Z}_{>0}$ such that $\dim_{\mathbb{F}_p} \mathcal{O}_K/\mathfrak{p}_i = f_i$ and

$$\sum_{i=1}^g e_i f_i = d.$$

Note that we are only interested in *unramified* primes p where all e_i 's are equal to 1, or else we would have to compute the determinant over $\mathcal{O}_K/\mathfrak{p}^{e_i}$ a ring containing zero divisors. In this unramified case we also have $\sum_{i=1}^g f_i = d$. By another famous theorem from algebraic number theory, see [6, Theorem 4.8.8.], we know that the primes not dividing Δ_K are exactly the unramified primes.

On the other hand we need to restrict ourselves to rational primes p not dividing $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$ since then we can efficiently compute the decomposition and the residue fields. This is due to the following theorem of Dedekind-Kummer, see [6, Theorem 4.8.13.]. Recall that f is the defining polynomial of the number field K chosen as in our assumptions.

Proposition 20. *Let p be a rational prime not dividing $[\mathcal{O} : \mathbb{Z}[\alpha]]$ and $\bar{f} = \prod_{i=1}^g \bar{f}_i^{e_i}$ the factorization of $\bar{f} \in \mathbb{F}_p[X]$ into irreducible polynomials. Then*

$$\mathcal{O}_K/p\mathcal{O}_K \cong \mathbb{Z}[\alpha]/p\mathbb{Z}[\alpha] \cong \mathbb{F}_p[X]/(\bar{f}) \cong \prod_{i=1}^g \mathbb{F}_p[X]/(\bar{f}_i^{e_i}).$$

Computing the factorization of (p) in \mathcal{O}_K is therefore equivalent to a polynomial factorization over a finite field. We now describe the complexity of passing to the residue field and of working in it. Assume that p is a fixed rational prime, unramified and not dividing $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$. The first task is the factorization of f modulo p which can be achieved by the deterministic algorithm of Shoup [20, Theorem 3.1.].

Proposition 21. *Let $p \in \mathbb{Z}_{>0}$ be a rational prime. The number of \mathbb{F}_p operations needed to compute the factorization of $\bar{f} \in \mathbb{F}_p[X]$ into irreducible polynomials is in $\tilde{O}(p^{1/2} \log(p)^2 d^2)$. Thus this has complexity in $\tilde{O}(p^{1/2} d^2 \log(p)^3)$.*

For each irreducible factor $\bar{f}_i \in \mathbb{F}_p[X]$ of \bar{f} we obtain the diagram

$$\mathcal{O}_K \longrightarrow \mathcal{O}_K/(p) \xrightarrow{\pi} \mathbb{F}_p[X]/(\bar{f}) \xrightarrow{\pi_i} \mathbb{F}_p[X]/(\bar{f}_i),$$

where π and π_i are the corresponding projections. We now determine the complexity of passing from \mathcal{O}_K to $\mathbb{F}_p[X]/(\bar{f}_i)$. Let $\beta = \sum_{i=1}^d b_i \omega_i$ be an integral element. Since π is a ring homomorphism we obtain

$$\pi(\alpha) = \sum_{j=1}^d \bar{a}_j \pi(\omega_j)$$

where $\bar{}$ denotes reduction $\mathbb{Z} \rightarrow \mathbb{F}_p$. Therefore we (only) need to evaluate π on the integral basis $(\omega_j)_j$. Denote by α the primitive element of K chosen in our assumption with minimal polynomial f . We consider the transformation matrix $M = (m_{ij})_{i,j} \in \mathbb{Z}^{d \times d}$ between the power basis $(\alpha^j)_{1 \leq j \leq d}$ and the integral basis Ω , which is defined by the equations

$$\alpha^j = \sum_{i=1}^d m_{ij} \omega_i$$

for $1 \leq j \leq d$. Then $\pi(\omega_i)$ is just the i -th column of $\overline{M}^{-1} \in \mathbb{F}_p^{d \times d}$, where $\overline{M} \in \mathbb{F}_p^{d \times d}$ is the matrix obtained by reducing each entry of M modulo p . For the complexity analysis we need a bound on the size of M . As $\alpha = \sum_{i=1}^d \varepsilon_i \omega_i$ with $\varepsilon_i \in \{0, 1\}$ we have $S(\alpha) = d$ and therefore $S(\alpha^j) \leq jS(\alpha) + jC \leq dS(\alpha) + dC$, that is, $\log(\|\alpha^j\|_\infty) \leq d + C$. This implies $\log(|M|) \leq C + d$ for the size of the entries of M .

Proposition 22. *Let $\overline{f}_1, \dots, \overline{f}_g, \pi$ and π_i and β as in the preceding discussion.*

1. *The $d \cdot g$ many images $\pi_i(\omega_j)$, $1 \leq j \leq d$, $1 \leq i \leq g$, can be computed with complexity in $\tilde{O}(d^3 \log(p) + d^2 C)$.*
2. *Let P be a set of primes. The reduction of the coefficient vector of β modulo all primes in P costs $\tilde{O}(d \sum_{l \in P} \log(l) + S(\beta))$.*
3. *Assuming that $\pi_i(\omega_j)$, $1 \leq j \leq d$, $1 \leq i \leq g$ as well as the reduction of the coefficient vector of β modulo p is known, the computation of $\pi_i(\beta)$, $1 \leq i \leq g$, has complexity in $\tilde{O}(d^2 \log(p))$.*

Proof. (1): The reduction of M modulo p has complexity in $\tilde{O}(d^2(\log(p) + C))$ and inverting the reduced matrix over the finite field \mathbb{F}_p has complexity in $\tilde{O}(d^3(\log(p)))$. Then for each $1 \leq j \leq d$ we have to reduce the elements $\pi_i(\omega_j)$ modulo \overline{f}_i for $1 \leq i \leq g$. By [20, Lemma 3.2] this has complexity in $d\tilde{O}(d \log(g)) \subseteq \tilde{O}(d^2)$.

(2): Using the remainder tree of Bernstein [2, 18.6] the computation for each coefficient is in $\tilde{O}(\sum_{l \in P} \log(l) + \log(\|\beta\|_\infty))$.

(3): We just have to compute d products $\overline{a}_j \pi_i(\omega_j)$, $1 \leq j \leq d$ and d additions of elements in $\mathbb{F}_p[X]/(\overline{f}_i)$. The last two steps have complexity in $\tilde{O}(d \deg(\overline{f}_i) \log(p))$. Thus summing over all $1 \leq i \leq g$ we obtain a complexity in $\tilde{O}(d^2 \log(p))$. \square

Working in the residue fields is just polynomial arithmetic over \mathbb{F}_p . For the sake of completeness we recall the necessary complexity, see for example [20, Lemma 3.2].

Remark 23. 1. *Let $a, b \in \mathbb{F}_p$ and $\star \in \{+, -, \cdot, \div\}$. The complexity of $a \star b$ is in $\tilde{O}(\log(p))$*

2. *Multiplication of two polynomials of degree $\leq d$ in $\mathbb{F}_p[X]$ can be performed using $\tilde{O}(d)$ operations in \mathbb{F}_p .*
3. *Let $f, g \in \mathbb{F}_p[X]$ be two polynomials of degree $\leq d$. Then $f \bmod g$ can be computed using $\tilde{O}(d)$ operations in \mathbb{F}_p . The greatest common divisor of f and g can be computed using $\tilde{O}(d)$ operations in \mathbb{F}_p .*
4. *Let $h \in \mathbb{F}_p[X]$ be a polynomial of degree bounded by d . Assume we have $\overline{g}, \overline{f} \in \mathbb{F}_p[X]/(h)$ and $\star \in \{+, -, \cdot, \div\}$. Then $\overline{g} \star \overline{f}$ can be computed using $\tilde{O}(d)$ operations in \mathbb{F}_p . Therefore each operation in $\mathbb{F}_p[X]/(h)$ has complexity in $\tilde{O}(d \log(p))$.*

Finally we describe how to combine the computations in the finite fields to obtain a result in $\mathcal{O}_K/(N)$.

Assume that we have a set P of rational primes and $N = \prod_{p \in P} p$. For each prime p we have a factorization of f modulo p into irreducible factors $\bar{f}_i \in \mathbb{F}_p[X]$, $1 \leq i \leq g$. Using the Chinese remainder theorem for polynomials we can construct a preimage under the map

$$\mathbb{F}_p[X]/(\bar{f}) \rightarrow \prod_{i=1}^g \mathbb{F}_p[X]/(\bar{f}_i).$$

The next step is an application of the Chinese remainder theorem for rational integers for each coefficient yielding a preimage under the map

$$\mathbb{Z}[X]/(\bar{f}, N) \longrightarrow \prod_{p \in P} \mathbb{F}_p[X]/(\bar{f}).$$

Finally we have to compute a preimage under the map $\mathcal{O}_K/(N) \rightarrow \mathbb{Z}[X]/(\bar{f}, N)$.

Proposition 24. *Using the notation from the preceding paragraph the following holds:*

1. Let $\bar{h}_i \in \mathbb{F}_p[X]/(\bar{f}_i)$ for $1 \leq i \leq d$. Computing $\bar{h} \in \mathbb{F}_p[X]/(\bar{f})$ such that $\pi_i(\bar{h}) = \bar{h}_i$, $1 \leq i \leq g$ has complexity in $\tilde{O}(d \log(p))$.
2. Assume we are given $\bar{g}_p \in \mathbb{F}_p[X]/(\bar{f})$ for $p \in P$. Then we can compute $\bar{h} \in \mathbb{Z}[X]/(f, N)$ with $\bar{h} = \bar{g}_p$ in $\mathbb{F}_p[X]/(\bar{f})$ for $p \in P$ with complexity in $\tilde{O}(d \log(B)r)$ where $B \in \mathbb{R}_{\geq 0}$ is such that $p \leq B$ for all $p \in P$ and $|P| = r \geq 2$ is the number of involved primes.
3. Given $\bar{g} \in \mathbb{Z}[X]/(f, N)$ the computation of a preimage under the map $\mathcal{O}_K/(N) \rightarrow \mathbb{Z}[X]/(f, N)$ has complexity in $\tilde{O}(d^2(d + C + \log(N)))$.

Proof. (1): This is Corollary 10.23 in [24].

(2): Due to Bernstein [2, §23] Chinese remaindering involving r moduli of size bounded by B has complexity in $\tilde{O}(\log(B)r)$. Since we have d coefficients, the result follows.

(3): This is just a matrix vector product between the coefficients of g and M . \square

On the number, choice and size of primes.

We still need to describe how many primes we need and of which size they are. By Theorem 18 the number $B = n^n C_1 C_2^n |A|^n \in \mathbb{R}_{>0}$ satisfies $\|\det(A)\|_\infty \leq B$. Choosing the first $r' = \lceil \log(B) \rceil$ primes we obtain $\prod_{i=1}^{r'} p_i > B/2$. As we have seen there is a finite number of bad primes we need to avoid. More precisely we are only interested in primes not dividing Δ_K and $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$. As Δ_K and $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$ have at most $\log(|\Delta_K|) + \log([\mathcal{O}_K : \mathbb{Z}[\alpha]])$ prime factors we see that the set of first $r = \log(B) + \log(|\Delta_K|) + \log([\mathcal{O}_K : \mathbb{Z}[\alpha]])$ primes P' contains a subset P such that $\prod_{p \in P} p > B$ and no element of P divides Δ_K or $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$. Here we have used that $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$ divides $|\text{disc}(f)|$.

We have the following classical result about the computation and size of the first r primes. It is an application of the detailed analysis of Rosser and Schoenfeld [18] as well as the sieve of Eratosthenes and can be found in [24, Theorem 18.10].

Proposition 25. *Let $r \in \mathbb{Z}_{>0}$. The first r prime numbers $p_1, \dots, p_r \in \mathbb{Z}_{>0}$ can be computed with complexity in $O(r(\log(r))^2 \log \log(r))$ and if $r \geq 2$ each prime satisfies $p_i \leq 2r \ln(r)$, that is, $\log(p_i) \in \tilde{O}(\log(r))$.*

The algorithm and its complexity.

In the preceding sections we have collected all the tools that we need to describe and analyze the determinant computation.

Algorithm 3 Determinant computation over \mathcal{O}_K

Input: $A \in \mathcal{O}_K^{n \times n}$,

Output: $\det(A)$.

- 1: Compute a bound $B \in \mathbb{R}_{>0}$ on $\|\det(A)\|_\infty$ and set $r = \lceil \log(B) + \log(\Delta_K) + \log(|\text{disc}(f)|) \rceil \in \mathbb{Z}_{>0}$.
 - 2: Compute the first r primes and choose r' many $P = \{p_1, \dots, p_{r'}\}$ not dividing Δ_K and $|\text{disc}(f)|$.
 - 3: **for** $p \in P$ **do**
 - 4: Compute $\bar{f}_1, \dots, \bar{f}_g \in \mathbb{F}_p[X]$ such that $\bar{f} = \bar{f}_1 \cdots \bar{f}_g$.
 - 5: Compute $\pi_j(\omega_i)$ for $1 \leq i \leq d$ and $1 \leq j \leq g$.
 - 6: Compute $\pi_j(A) \in (\mathbb{F}_p[X]/(\bar{f}_j))^{n \times n}$ for $1 \leq j \leq g$.
 - 7: Compute $d_j = \det(\pi_j(A)) \in \mathbb{F}_p[X]/(\bar{f}_j)$ for $1 \leq j \leq g$.
 - 8: Compute $\bar{g}_p \in \mathbb{F}_p[X]/(f)$ such that $g_p = d_j$ in $\mathbb{F}_p[X]/(\bar{f}_j)$ for $1 \leq j \leq g$.
 - 9: **end for**
 - 10: Compute $\bar{g} \in \mathbb{Z}[X]/(f, N)$ such that $\bar{g} = g_p$ in $\mathbb{F}_p[X]/(\bar{f})$ for all $p \in P$.
 - 11: Compute the image of \bar{g} under $\mathbb{Z}[X]/(f, N) \rightarrow \mathcal{O}_K/(N)$ where $N = \prod_{p \in P} p$.
 - 12: **return** X .
-

Theorem 26. *Algorithm 3 is correct and has complexity in*

$$\tilde{O}(d^2 r^{3/2} + r(d^2 C + d^3 + dn^3 + d^2 n^2)),$$

where $r = n \log(|A|) + \log(|\Delta_K|) + nC$.

Proof. The correctness follows from the preceding paragraphs. By Proposition 25 Step 2 costs $\tilde{O}(r)$ and every $p \in P$ satisfies $p \leq 2r \ln(r)$. As $\log(p) \in \tilde{O}(\log(r)) = \tilde{O}(1)$ we will ignore all polynomial terms in $\log(p)$. Let us now consider the loop in Steps 3–9 excluding Step 5. As already noticed the factorization of f modulo p has complexity in $\tilde{O}(p^{1/2} d^2 + dC) \subseteq \tilde{O}(r^{1/2} d^2 + dC)$. By Proposition 22 computing the image of $(\omega_i)_i$ under the various π_j has complexity in $\tilde{O}(d^3 + d^2 C)$. Each determinant computation consists of $O(n^3)$ operations

in $\mathbb{F}_p[X]/(\bar{f}_i)$ taking $\tilde{O}(n^3 \deg(\bar{f}_i))$ bit operations in total. Consequently by summing over all $1 \leq i \leq g$ we see that Step 7 has complexity in $\tilde{O}(dn^3)$. By virtue of Proposition 24 Step 8 has complexity in $\tilde{O}(d)$. Since these steps are repeated r times we obtain a complexity in $\tilde{O}(r(d^2r^{1/2} + d^2C + d^3 + dn^3))$. Now consider the missing Step 5. Reducing the coefficients of all entries of A modulo all primes $p \in P$ has complexity in $\tilde{O}(dn^2 \sum_{p \in P} \log(p) + dn^2 \log(|A|)) \subseteq \tilde{O}(dn^2r + dn^2 \log(|A|))$ by Proposition 22 (2). To compute $\pi_j(A)$ we apply item (3) of the same proposition and arrive at a complexity of $\tilde{O}(d^2n^2r)$ since we have to do it r times. In total the inner loop in Steps 3–9 has a complexity in

$$\tilde{O}(r(d^2r^{1/2} + d^2C + d^3 + dn^3 + d^2n^2) + dn^2 \log(|A|)).$$

As Steps 10 and 11 have complexity in $\tilde{O}(dr)$ and $\tilde{O}(d^2(C + d + \log(N))) \subseteq \tilde{O}(d^2(C + d + r))$ respectively, we get an overall complexity in

$$\tilde{O}(r(d^2r^{1/2} + d^2C + d^3 + dn^3 + d^2n^2) + dn^2 \log(|A|) + d^2C + d^3 + d^2r).$$

Finally we use the fact that $r \in O(\log(B) + \log(|\text{disc}(f)|) + \log(|\Delta_K|)) \subseteq \tilde{O}(n \log(|A|) + nC + \log(|\Delta_K|))$ to conclude that the complexity of Algorithm 3 is in $\tilde{O}(d^2r^{3/2} + r(d^2C + d^3 + dn^3 + d^2n^2))$. \square

In case we fix the number field K , that is, we ignore the constants coming from field arithmetic, the complexity of Algorithm 3 reduces to $\tilde{O}((n \log(|A|))^{3/2} + n^4 \log(|A|))$.

Note that in contrast to the integer case our algorithm is not softly linear in $\log(|A|)$ which can be explained as follows: Recall that our small primes approach needs at least $\log(|A|)$ primes which are roughly of the same order as $\log(|A|)$. As the deterministic factorization in \mathbb{F}_p has costs $\tilde{O}(p^{1/2})$ (ignoring the dependency on the degree), the complexity of all factorizations contains at least a factor of $\log(|A|) \log(|A|)^{1/2} = \log(|A|)^{3/2}$. Consequently we see that the exponential factorization algorithm is the bottleneck of our determinant algorithm. While there exist various probabilistic polynomial algorithms for the factorization over \mathbb{F}_p , they are unusable for us, since we are aiming at an deterministic polynomial pseudo-HNF algorithm. We can now address the problem of computing the determinantal ideal.

Corollary 27. *Assume that $M = (A, (\mathfrak{a}_i)_i)$ is a pseudo-matrix with $A \in \mathcal{O}_K^{n \times n}$. There exists a deterministic algorithm computing*

$$\mathfrak{d}(M) = \det(A) \prod_{i=1}^n \mathfrak{a}_i$$

with complexity in

$$\tilde{O}(d^2r^{3/2} + r(d^2C + d^3 + dn^3 + d^2n^2) + d^2nB + d^4nC),$$

where $r = n \log(|A|) + \log(|\Delta_K|) + nC$ and $B = \max_i S(\mathfrak{a}_i)$.

Proof. It remains to evaluate the complexity of the ideal product. As a divide and conquer approach shows that the product can be computed with complexity in $\tilde{O}(d^2n \log(n)B + d^4nC)$ the claim follows. \square

The rectangular case.

The case where the pseudo-matrix is not square is more involved. We will now describe an algorithm for computing an integral multiple of the determinantal ideal of a pseudo-matrix.

Theorem 28. *There exists a deterministic algorithm that given a matrix $A \in \mathcal{O}_K^{n \times m}$, $m \leq n$, returns the rank s of A , a non-singular $s \times s$ submatrix A' of A and $\det(A')$. The algorithm has complexity in*

$$\tilde{O}(d^2 r^{3/2} + r(d^2 C + d^3 + dnm^2 + d^2 nm)),$$

where $r = m \log(|A|) + \log(|\Delta_K|) + mC$.

Proof. Let s be the rank of A and $A' \in \mathcal{O}_K^{s \times s}$ a non-singular submatrix of A . Using Lemma 18 we see that $\log(\|\det(A')\|_\infty) \in O(s \log(s|A'|) + sC) \subseteq O(m \log(m|A|) + mC)$. Now let $B \in \mathbb{R}_{>0}$ be a number with $\log(B) \in O(m \log(m|A|) + mC)$ such that B exceeds $2 \|\det(A')\|_\infty$ for all non-singular $s \times s$ submatrices of A , and let $\mathfrak{p}_1, \dots, \mathfrak{p}_l$ be coprime prime ideals of \mathcal{O}_K such that $B \in \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_l$. If now A' is a non-singular $s \times s$ submatrix of A there exists i such that $A' \bmod \mathfrak{p}_i$ has non-zero determinant and consequently $A \bmod \mathfrak{p}_i$ has rank s . For if this is not the case, we would have $\det(A') \equiv 0 \bmod \mathfrak{p}_1 \cdots \mathfrak{p}_l$ yielding $\det(A') \equiv 0 \bmod (B)$ and $\det(A') = 0$ by Lemma 19, a contradiction.

Thus to find the rank we choose a set of primes P such that $\prod_{p \in P} p \geq B$ and all $p \in P$ are unramified and do not divide $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$. For every prime $p \in P$ we compute the prime ideal factorization of $p\mathcal{O}_K$ and for all these prime ideals we compute the rank of $A \bmod \mathfrak{p}$ using Gaussian elimination. Similar to Algorithm 3 this has complexity in $\tilde{O}(d^2 r^{3/2} + r(d^2 C + d^3 + dnm^2 + d^2 nm))$, where $r = m \log(|A|) + \log(|\Delta_K|) + mC$.

Let \mathfrak{p} be a prime ideal lying above one of the $p \in P$ such that $A \bmod \mathfrak{p}$ has maximal rank. Then the rank of $A \bmod \mathfrak{p}$ is the rank s of A and we can find a non-singular $s \times s$ submatrix A' of A . The computation of $\det(A')$ using Algorithm 3 has complexity in

$$\tilde{O}(d^2 r^{3/2} + r(d^2 C + d^3 + ds^3 + d^2 s^2)) \subseteq \tilde{O}(d^2 r^{3/2} + r(d^2 C + d^3 + dnm^2 + d^2 nm)).$$

□

Combining this result with Corollary 27 immediately yields:

Corollary 29. *Assume that $M = (A, (\mathfrak{a}_i)_i)$ is a pseudo-matrix with $A \in \mathcal{O}_K^{n \times m}$, $n \geq m$, of rank m . There exists a deterministic algorithm computing a multiple of $\mathfrak{d}(M)$ with complexity in*

$$\tilde{O}(d^2 r^{3/2} + r(d^2 C + d^3 + dnm^2 + d^2 nm) + d^2 mB + d^4 mC),$$

where $r = m \log(|A|) + \log(|\Delta_K|) + mC$ and $B = \max_i S(\mathfrak{a}_i)$.

6. The pseudo-HNF algorithm

Constructing idempotents.

In order to compute the pseudo-HNF over Dedekind domains, we use the constructive version of the Chinese remainder theorem introduced by Cohen in [7]. More precisely given coprime integral ideals \mathfrak{a} and \mathfrak{b} of \mathcal{O}_K , we need to find $\alpha \in \mathcal{O}_K$ such that $\alpha \in \mathfrak{a}$ and $1 - \alpha \in \mathfrak{b}$. This problem is closely connected to the computation of the sum of \mathfrak{a} and \mathfrak{b} : The HNF of the matrix

$$A = \left(\begin{array}{c|c} M_{\mathfrak{a}} & M_{\mathfrak{a}} \\ \hline \mathbf{0} & M_{\mathfrak{b}} \end{array} \right)$$

is equal to

$$\left(\begin{array}{c|c} * & \mathbf{0} \\ \hline U & M_{\mathfrak{a}+\mathfrak{b}} \end{array} \right) = \left(\begin{array}{c|c} * & \mathbf{0} \\ \hline U & \mathbf{1}_d \end{array} \right)$$

for some $U \in \mathbb{Z}^{d \times d}$ since $\mathfrak{a} + \mathfrak{b} = \mathcal{O}_K$. Denoting by $v \in \mathbb{Z}^d$ be the first row of U we see that the element $\alpha = \sum_{i=1}^d v_i \omega_i$ of \mathcal{O}_K satisfies $\alpha \in \mathfrak{a}$ and $1 - \alpha \in \mathfrak{b}$.

Lemma 30. *Given coprime integral ideals $\mathfrak{a}, \mathfrak{b}$ of \mathcal{O}_K , there exists a deterministic algorithm which computes elements $\alpha \in \mathfrak{a}$ and $\beta \in \mathfrak{b}$ such that $\alpha + \beta = 1$. Moreover the output satisfies $S(\alpha), S(\beta) \in \tilde{O}((S(\mathfrak{a}) + S(\mathfrak{b}))/d)$ and the complexity of the algorithm is in*

$$\tilde{O}(d(S(\mathfrak{a}) + S(\mathfrak{b}))).$$

Proof. We use the same notation as in the preceding discussion. Note that $\lambda = \min(\mathfrak{a}) \min(\mathfrak{b})$ satisfies $\lambda \mathbb{Z}^{2d} \in [A]_{\mathbb{Z}}$ allowing us to compute the HNF with complexity in $\tilde{O}(d^3 \log(\min(\mathfrak{a}) \min(\mathfrak{b}))) = \tilde{O}(d(S(\mathfrak{a}) + S(\mathfrak{b})))$. Moreover as $\log(|U|) \leq 2 \log(\lambda)$ we know that $S(\alpha) = d \log(|v|) \in O((S(\mathfrak{a}) + S(\mathfrak{b}))/d)$. \square

Using this construction we can now describe an algorithm, which plays the same role as the extended GCD algorithm over the integers. It is the workhorse of the pseudo-HNF algorithm and is accompanied by the normalization and reduction procedures which provide bounded input.

Algorithm 4 Euclidean step

Input: Fractional ideals $\mathfrak{a}, \mathfrak{b}$ and elements $\alpha, \beta \in K$.

Output: $\mathfrak{g} = \alpha \mathfrak{a} + \beta \mathfrak{b}$, \mathfrak{g}^{-1} and $\gamma \in \mathfrak{a} \mathfrak{g}^{-1}$ and $\delta \in \mathfrak{b} \mathfrak{g}^{-1}$ such that $\alpha \gamma + \beta \delta = 1$.

- 1: Compute $\mathfrak{g} = \alpha \mathfrak{a} + \beta \mathfrak{b}$, \mathfrak{g}^{-1} , $\mathfrak{a} \mathfrak{g}^{-1}$ and $\mathfrak{b} \mathfrak{g}^{-1}$.
 - 2: Apply Lemma 30 to $\alpha \mathfrak{a} \mathfrak{g}^{-1}$ and $\beta \mathfrak{b} \mathfrak{g}^{-1}$ and denote the output by $\tilde{\gamma}, \tilde{\delta}$.
 - 3: **return** $\gamma = \tilde{\gamma} \alpha^{-1}$ and $\delta = \tilde{\delta} \beta^{-1}$.
-

Proposition 31. *Algorithm 4 is correct and has complexity in*

$$\tilde{O}(d^2(S(\mathfrak{a}) + S(\mathfrak{b})) + d^4(S(\alpha) + S(\beta)) + d^3 C + d^3 \log(|\Delta_K|)).$$

The output satisfies $S(\gamma), S(\delta) \in \tilde{O}((S(\mathfrak{a}) + S(\mathfrak{b}))/d + d(S(\alpha) + S(\beta)) + C)$.

Proof. Correctness is clear. The first step consists of the computation of $\alpha\mathfrak{a}$ and $\beta\mathfrak{b}$, which has complexity in $\tilde{O}(d^3(\mathsf{S}(\alpha) + \mathsf{S}(\beta)) + d(\mathsf{S}(\mathfrak{a}) + \mathsf{S}(\mathfrak{b})) + d^2C)$. Denote by B the value $\mathsf{S}(\alpha\mathfrak{a}) + \mathsf{S}(\beta\mathfrak{b}) \in O(\mathsf{S}(\mathfrak{a}) + \mathsf{S}(\mathfrak{b}) + d^2\mathsf{S}(\alpha) + d^2\mathsf{S}(\beta) + dC)$. While the computation of \mathfrak{g} has complexity in $\tilde{O}(dB)$ the inversion costs $\tilde{O}(dB + d^3 \log(|\Delta_K|) + d^3C)$. As $\mathsf{S}(\mathfrak{g}) \in O(B)$ the inverse ideal \mathfrak{g}^{-1} also satisfies $\mathsf{S}(\mathfrak{g}^{-1}) \in O(B)$. Finding the product $\beta\mathfrak{b}\mathfrak{g}^{-1}$ and $\alpha\mathfrak{a}\mathfrak{g}^{-1}$ then has complexity in $\tilde{O}(d^2B + d^3C)$ and the size of both integral ideals is in $\tilde{O}(B)$. Hence invoking Lemma 30 has a complexity in $\tilde{O}(dB)$ and the resulting elements satisfy $\mathsf{S}(\tilde{\gamma}), \mathsf{S}(\tilde{\delta}) \in \tilde{O}(B/d)$. Finally we have to compute inverses and products. While α^{-1} and β^{-1} can be computed in $\tilde{O}(d^2\mathsf{S}(\alpha) + d^2\mathsf{S}(\beta) + d^2C)$ the costs of the products are in $\tilde{O}(d^2\mathsf{S}(\tilde{\gamma}) + d^2\mathsf{S}(\tilde{\delta}) + d^2\mathsf{S}(\alpha) + d^2\mathsf{S}(\beta) + d^2C)$. Thus the ideal product dominates the complexity of the algorithm and the claim follows. Note that $\mathsf{S}(\gamma) = \mathsf{S}(\tilde{\gamma}\alpha^{-1}) \leq \mathsf{S}(\tilde{\gamma}) + d\mathsf{S}(\alpha) + C \in \tilde{O}(B/d + d(\mathsf{S}(\alpha)))$ and a similar result holds for $\mathsf{S}(\delta)$. \square

The main algorithm and its complexity.

Assume that $M \subseteq \mathcal{O}_K^m$ is an \mathcal{O}_K -module of rank m given by a pseudo-basis $(A, (\mathfrak{a}_i)_i)$ with $A \in K^{n \times m}$ ($n \geq m$). Using this input we now describe a polynomial time algorithm for computing the pseudo-HNF of M . The algorithm is a variant of the so-called modular algorithm of Cohen, the big difference being the normalization of the coefficient ideals. Using this extra feature we are able bound the denominators of the coefficients of the matrix. Together with the reduction procedure this will allow us to prove polynomial running time. Invoking Theorem 28 we may assume that we know the determinantal ideal \mathfrak{d} of M . This case often occurs, for example when computing with ideals in relative extension.

First of all, we want to show that at the beginning of the inner loop at Step 6 the sizes of B_i, B_j and $\mathfrak{b}_i, \mathfrak{b}_j$ are bounded. We use an inductive argument and begin with the size of the objects at Step 3. Let $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. As the ideal \mathfrak{b}_i is normalized it satisfies

$$\min(\mathfrak{b}_i) \leq \mathsf{N}(\mathfrak{b}_i) \leq \ell^{d^2} \sqrt{|\Delta_K|}.$$

By Proposition 15 the reduction of Step 2 yields

$$\|\beta_{i,j}\| \leq d^{3/2} \ell^{d^2} \mathsf{N}(\mathfrak{d}\mathfrak{b}_i^{-1})^{1/d} \sqrt{|\Delta_K|} \leq d^{3/2} \ell^{d^2} \min(\mathfrak{d}) \sqrt{|\Delta_K|}.$$

As $\beta_{i,j}\mathfrak{b}_i \subseteq \mathcal{O}_K$ the denominator $l \in \mathbb{Z}_{>0}$ of $\beta_{i,j}$ satisfies $l \leq \min(\mathfrak{b}_i)$; in particular

$$\begin{aligned} \mathsf{S}(\beta_{i,j}) &= d \log(\|l\beta_{i,j}\|_\infty) + d \log(l) \\ &= 2d \log(l) + d \log(\|\beta_{i,j}\|_\infty) \in \tilde{O}(\mathsf{S}(\mathfrak{d})/d + \mathsf{S}(\mathfrak{b}_i)/d + C). \end{aligned}$$

We define $B_{\text{id}} = d^4 + d^2 \log(|\Delta_K|)$ and $B_e = \mathsf{S}(\mathfrak{d})/d + B_{\text{id}}/d + C$ respectively. The inequalities $B_{\text{id}} \leq dB_e$ and $C + d \log(|\Delta_K|) + d^3 \leq B_e$ will be used throughout the following complexity analysis.

Algorithm 5 pseudo-HNF of a full-rank pseudo-matrix

Input: Full-rank pseudo-matrix $(A, (\mathfrak{a}_i)_i)$ of an \mathcal{O}_K -module $M \subseteq \mathcal{O}_K^m$ and $\mathfrak{d} = \det(M)$.

Output: Pseudo-HNF $(B, (\mathfrak{b}_i)_i)$ of M .

- 1: Let $(B, (\mathfrak{b}_i)_i) = (A, (\mathfrak{a}_i)_i)$. Normalize $(A_i, \mathfrak{a}_i)_{1 \leq i \leq n}$ with Algorithm 2.
 - 2: Reduce A_i modulo $\mathfrak{d}\mathfrak{a}_i^{-1}$ using Algorithm 1 for $1 \leq i \leq n$.
 - 3: $\mathfrak{D} \leftarrow \mathfrak{d}$.
 - 4: **for** $i = n, \dots, n - m + 1$ **do**
 - 5: **for** $j = i - 1, \dots, 1$ **do**
 - 6: $\mathfrak{g} \leftarrow \beta_{j,i}\mathfrak{b}_j + \beta_{i,i}\mathfrak{b}_i$
 - 7: If $\mathfrak{g} = 0$ go to step 5.
 - 8: Compute $\gamma \in \mathfrak{b}_j\mathfrak{g}^{-1}$ and $\delta \in \mathfrak{b}_i\mathfrak{g}^{-1}$ such that $\beta_{j,i}\gamma + \beta_{i,i}\delta = 1$ using Algorithm 4.
 - 9: $(\mathfrak{b}_j, \mathfrak{b}_i) \leftarrow (\mathfrak{b}_j\mathfrak{b}_i\mathfrak{g}^{-1}, \mathfrak{g})$.
 - 10: $(B_j, B_i) \leftarrow (\beta_{i,i}B_j - \beta_{j,i}B_j, \gamma B_j + \delta B_i)$.
 - 11: Normalize (B_j, \mathfrak{b}_j) and (B_i, \mathfrak{b}_i) using Algorithm 2
 - 12: Reduce B_j modulo $\mathfrak{d}\mathfrak{b}_j^{-1}$ and B_i modulo $\mathfrak{d}\mathfrak{b}_i^{-1}$ using Algorithm 1.
 - 13: **end for**
 - 14: Set $\mathfrak{g} = \beta_{i,i}\mathfrak{b}_i + \mathfrak{D}$. Compute $\gamma \in \mathfrak{b}_i\mathfrak{g}^{-1}$ and $\delta \in \mathfrak{D}\mathfrak{g}^{-1}$ such that $\gamma\beta_{i,i} + \delta = 1$.
 - 15: Set $B_i \leftarrow \gamma B_i \bmod \mathfrak{D}\mathfrak{g}^{-1}$ using Algorithm 1 and $\mathfrak{b}_i \leftarrow \mathfrak{g}$, $\beta_{i,i} \leftarrow 1$.
 - 16: $\mathfrak{D} \leftarrow \mathfrak{D}\mathfrak{g}^{-1}$.
 - 17: **end for**
 - 18: Move all non-zero rows, together with their coefficient ideals, to the top of B .
 - 19: **return** $(B, (\mathfrak{b}_i)_i)$.
-

Proposition 32. Let $n - m + 1 \leq i \leq n$ and $1 \leq j \leq i - 1$. At the beginning of the inner loop at Step 7 the size of the coefficient ideals $\mathfrak{b}_i, \mathfrak{b}_j$ is bounded by B_{id} and the size of the elements of rows B_i, B_j is in $\tilde{O}(B_e)$.

Proof. This follows from Steps 10 and 11. \square

We will now analyze the complexity of the algorithm. In order to improve readability we split up the analysis according to the single steps. Let us first take care of the steps in the loops.

Lemma 33. Let $(A, (\mathfrak{a}_i)_i)$ be as in the input of Algorithm 5.

1. Steps 6–7 have complexity in $\tilde{O}(d^4 B_e)$.
2. Step 8 has complexity in $\tilde{O}(d^4 B_e)$.
3. Step 9 has complexity in $\tilde{O}(d^2(d + m)B_e)$.
4. Step 10 has complexity in $\tilde{O}(d^5 B_e + d^3 m B_e)$.
5. Step 11 has complexity in $\tilde{O}(d^3 \mathfrak{S}(\mathfrak{d}) + dm\mathfrak{S}(\mathfrak{d}) + d^4 m B_e)$.
6. Step 13 has complexity in $\tilde{O}(d^2 \mathfrak{S}(\mathfrak{d}) + d^4 B_e)$.
7. Step 14 has complexity in $\tilde{O}(d^3 \mathfrak{S}(\mathfrak{d}) + dm\mathfrak{S}(\mathfrak{d}) + d^3 m B_e)$.

Thus the inner loop in Steps 6–11 as well as Steps 13–15 is dominated by normalization and reduction yielding an overall complexity in

$$\tilde{O}(d^3(d+m)(S(\mathfrak{d}) + d^4 + d^2 \log(|\Delta_K|) + dC)).$$

Proof. (1): Steps 6–7 are just an application of Algorithm 4 with complexity in $\tilde{O}(d^2 B_{\text{id}} + d^4 B_e + d^3 C + d^3 \log(|\Delta_K|)) \subseteq \tilde{O}(d^4 B_e)$. The size of γ and δ is in $\tilde{O}(B_{\text{id}}/d + dB_e + C) \subseteq \tilde{O}(dB_e)$.

(2): The size of \mathfrak{g} and therefore also the size of \mathfrak{g}^{-1} is in $\tilde{O}(B_{\text{id}} + d^2 B_e + dC) \subseteq \tilde{O}(d^2 B_e)$. As we have already computed \mathfrak{g}^{-1} in Algorithm 4, the computation of $\mathfrak{b}_i \mathfrak{b}_j \mathfrak{g}^{-1}$ has complexity in $\tilde{O}(d^2 B_{\text{id}} + d^2(d^2 B_e) + d^3 C) \subseteq \tilde{O}(d^4 B_e)$. Note that $S(\mathfrak{b}_i \mathfrak{b}_j \mathfrak{g}^{-1}) \in \tilde{O}(B_{\text{id}} + d^2 B_e) \subseteq \tilde{O}(d^2 B_e)$.

(3): Since $S(\gamma), S(\delta) \in \tilde{O}(dB_e)$, computing the scalar vector products has complexity in $\tilde{O}(d(d+m)(dB_e) + dmB_e + d(d+m)C) \subseteq \tilde{O}(d^2(d+m)B_e)$. The size of the new elements in row i and j is in $\tilde{O}(dB_e)$.

(4): The normalization has complexity in $\tilde{O}(d(d^2 + m)(d^2 B_e) + dm(dB_e) + d(d^2 + m)(\log(|\Delta_K|) + C))$ which simplifies to $\tilde{O}(d^5 B_e + d^3 m B_e)$. While by definition the new ideals have size bounded by B_{id} , the size of the new elements is in $\tilde{O}(d^2 B_e + dB_e + d \log(|\Delta_K|) + dC) = \tilde{O}(d^2 B_e)$.

(5): Inverting \mathfrak{b}_i and \mathfrak{b}_j has complexity in $\tilde{O}(dB_{\text{id}} + d^3 \log(|\Delta_K|) + d^2 C)$ and the multiplication with \mathfrak{d} is in $\tilde{O}(d^2(B_{\text{id}} + S(\mathfrak{d})) + d^3 C)$. The reduction itself then has complexity in $\tilde{O}(d(d^2 + m)(B_{\text{id}} + S(\mathfrak{d})) + d^2 m(d^2 B_e) + d^2(d+m)C + d^3 m \log(|\Delta_K|))$ which is in $\tilde{O}(d^3 S(\mathfrak{d}) + dmS(\mathfrak{d}) + d^4 m B_e)$.

(6): Step 13 is again an application of Algorithm 4 with complexity in $\tilde{O}(d^2(B_{\text{id}} + S(\mathfrak{d})) + d^4 B_e + d^3 C + d^3 \log(|\Delta_K|)) \subseteq \tilde{O}(d^2 S(\mathfrak{d}) + d^4 B_e)$ and again the size of γ and δ is in $\tilde{O}(S(\mathfrak{d})/d + dB_e)$. Here we have used that $S(\mathfrak{d}) \leq S(\mathfrak{d})$ since \mathfrak{d} is a divisor of \mathfrak{d} .

(7): While the product $\mathfrak{d} \mathfrak{g}^{-1}$ was already computed in Algorithm 4, the computation of γB_i has complexity in $\tilde{O}(d(d+m)S(\gamma) + dmB_e + d(d+m)C) = \tilde{O}((d+m)S(\mathfrak{d}) + d^2(d+m)B_e)$. Since the entries of γB_i have size in $\tilde{O}(S(\mathfrak{d})/d + dB_e)$ the final reduction is in $\tilde{O}(d^3 S(\mathfrak{d}) + dmS(\mathfrak{d}) + d^2 m(S(\mathfrak{d})/d + dB_e) + d^2(d+m)C + d^3 m \log(|\Delta_K|))$ which simplifies to $\tilde{O}(d^3 S(\mathfrak{d}) + dmS(\mathfrak{d}) + d^3 m B_e)$. \square

Theorem 34. *Algorithm 5 is correct and the complexity is in*

$$\tilde{O}(d^2 n(d+m) \max_i S(\mathfrak{a}_i) + d^2 nm \max_{i,j} S(\alpha_{i,j}) + d^3 nm(d+m)(S(\mathfrak{d}) + d^4 + d^2 \log(|\Delta_K|) + dC)).$$

Proof. We first consider the correctness. Note that the only difference between our algorithm and [7, Algorithm 3.2] are the normalizations. As the normalizations do not change the module (see Proposition 17), we conclude that the correctness proof of [7, Algorithm 3.2] carries over.

We now turn to the complexity. Since the inner loop is executed $O(nm)$ times we conclude using Lemma 33 that Steps 5–18 have complexity in $\tilde{O}(d^3 nm(d+m)(S(\mathfrak{d}) + d^4 + d^2 \log(|\Delta_K|) + dC))$. Now we consider the initialization in Step 1–3. Denote $\max_{i,j} S(\alpha_{i,j})$ and $\max_i S(\mathfrak{a}_i)$ by B_A and B_a respectively. By Proposition 17 Step 1 has complexity in $\tilde{O}(dn(d^2 + m)B_a + dnmB_A + nd(d^2 +$

$m)(\log(|\Delta_K|)+C))$ and the new elements have size in $\tilde{O}(B_a+B_A+d\log(|\Delta_K|)+dC)$. As in the proof of Lemma 33, computing the product $\mathbf{b}_i^{-1}\mathfrak{d}$ has complexity in $\tilde{O}(dB_{\text{id}}+d^3\log(|\Delta_K|)+d^2(B_{\text{id}}+\mathbf{S}(\mathfrak{d}))+d^3C)$. Since this is repeated n times this has complexity in $\tilde{O}(d^2nB_{\text{id}}+d^2n\mathbf{S}(\mathfrak{d})+nd^3C)$. The reductions then cost $\tilde{O}(d(d^2+m)(B_{\text{id}}+\mathbf{S}(\mathfrak{d}))+d^2m(B_a+B_A+d\log(|\Delta_K|)+dC)+d^2(d+m)C+d^3m\log(|\Delta_K|))$ per row, i. e. $\tilde{O}(dn(d^2+m)\mathbf{S}(\mathfrak{d})+d^2nmB_A+d^2nmB_a)$ in total neglecting C and $\log(|\Delta_K|)$. Now the claim follows. \square

Remark 35. In [7] the following uniqueness result is proven: Let $M \subseteq \mathcal{O}_K^m$ be an \mathcal{O}_K -module with pseudo-HNF $(A, (\mathfrak{a}_i)_i)$. Then any other pseudo-HNF of M has the same coefficient ideals $(\mathfrak{a}_i)_i$. For i, j fix representatives $S_{i,j}$ for $K/\mathfrak{a}_i^{-1}\mathfrak{a}_j$. Then using suitable row operations, the pseudo-HNF $(A, (\mathfrak{a}_i)_i)$ can be transformed into a pseudo-HNF $(B, (\mathfrak{a}_i)_i)$, $B = (\beta_{i,j})_{i,j}$, of M such that $\beta_{i,j} \in S_{i,j}$ for all i, j . Moreover, $(B, (\mathfrak{a}_i)_i)$ is unique with this property.

The only difficult step is to find the sets of representatives. Let $(\alpha_1, \dots, \alpha_d)$ be the unique \mathbb{Z} -basis of $\mathfrak{a}_i^{-1}\mathfrak{a}_j$ obtained from the unique HNF basis of the numerator of $\mathfrak{a}_i^{-1}\mathfrak{a}_j$. Applying Algorithm 1 with $(\alpha_1, \dots, \alpha_d)$ instead of an LLL basis we can find for every $\alpha \in K$ an element $\tilde{\alpha}$ such that $\alpha - \tilde{\alpha} \in \mathfrak{a}_i^{-1}\mathfrak{a}_j$. Thus the representative $\tilde{\alpha}$ is defined to be the output of Algorithm 1 applied to α . Note that this yields unique representatives for each class since $\alpha - \beta \in \mathfrak{a}_i^{-1}\mathfrak{a}_j$ implies $\tilde{\alpha} = \tilde{\beta}$. In particular for the representatives $S_{i,j}$ we can just take $\{\tilde{\alpha} \mid \alpha \in K\}$.

Remark 36. The algorithm we described works only in the case that $M \subseteq \mathcal{O}_K^m$ has rank m , and cannot be applied in case the embedding dimension is higher than the rank. Moreover there seems to be no obvious adaption of the algorithm to the non-full rank case. For an important ingredient of the algorithm is the existence of an integral ideal \mathfrak{m} of \mathcal{O}_K with $\mathfrak{m}\mathcal{O}_K^m \subseteq M$, allowing us to reduce matrix entries modulo ideals (involving \mathfrak{m}). As the existence of such an ideal is equivalent to M being of full rank, any algorithm relying on this modular approach will have this restriction.

Remark 37. Although using lattice reduction in the normalization step we are able to bound the size of the coefficient ideals, the size already contains a factor d^4 . Together with the expensive ideal operations this explains the high dependency on d . In addition, the normalization and reduction steps themselves involve a costly lattice reduction algorithm. Unfortunately the dependency of the overall complexity of Algorithm 5 on the chosen lattice reduction algorithm is rather involved. We find ourselves on the horns of a dilemma – we have to make sure that the lattice reduction is not too expensive, but at the same time, we need small lattice bases to bound the size of elements and ideals during our algorithm.

Relative versus absolute computations

We now want to compare the pseudo-HNF algorithm with the HNF algorithm over the integers in situations where we can “choose” the structure we work with. We describe two examples to illustrate the idea.

In practice number fields of large degree are constructed carefully as towers of extensions of type $L \supseteq K \supseteq \mathbb{Q}$ where K is a number field of degree d and L is an extension of K of degree n . The ring of integers \mathcal{O}_L of L as well as the fractional ideals of L are naturally finitely generated modules of rank d over the Dedekind domain \mathcal{O}_K . On the other hand, \mathcal{O}_L as well as the fractional ideals of L are naturally free of rank dn over the principal ideal domain \mathbb{Z} . Thus the computation with ideals in \mathcal{O}_L can either rely on the pseudo-HNF over \mathcal{O}_K or on the HNF over \mathbb{Z} and it is not clear which to prefer.

The second situation we have in mind is quite different. Assume that we are in a situation where we have two finitely generated torsion free \mathcal{O}_K -modules M and N and we are faced with the problem of deciding $M \subseteq N$ and $M = N$. After imposing further properties on a pseudo-HNF yielding uniqueness the problem can be settled using the pseudo-HNF algorithm. But as the question only depends on the underlying sets of M and N (discarding the \mathcal{O}_K -structure) the problem can also be sorted out using the HNF over the integers. Again it is not clear which method to prefer.

We consider $(A, (\mathfrak{a}_i)_i)$, a full-rank pseudo-matrix over \mathcal{O}_K with $A \in K^{n \times n}$ and associated module $M \subseteq \mathcal{O}_K^n$. To compute the structure over the integers we have to turn this pseudo-matrix into a $dn \times dn$ matrix over the integers. As each fractional ideal \mathfrak{a}_i is isomorphic to \mathbb{Z}^d as a \mathbb{Z} -module, we have $M = A_1 \mathfrak{a}_1 + \dots + A_n \mathfrak{a}_n \cong \mathbb{Z}^{dn}$, the isomorphism being induced by the isomorphisms $\mathfrak{a}_i \rightarrow \mathbb{Z}^d$. Assume that $\beta \in K$ is an element of the i -th row of A and $\mathfrak{a} = \mathfrak{a}_i$ is the corresponding coefficient ideal of this row. Denote by $\alpha_1, \dots, \alpha_d$ the HNF basis of \mathfrak{a} . The coefficients of the d products $\beta \alpha_1, \dots, \beta \alpha_d$ form a $d \times d$ \mathbb{Z} -matrix. Applying this procedure to all matrix entries of A we obtain a $dn \times dn$ matrix B over the integers, which corresponds to a basis of the free \mathbb{Z} -module M of rank dn . These are n^2 computations each having complexity in $\tilde{O}(d^2 \max(\mathbf{S}(\alpha_{i,j})) + d \max(\mathbf{S}(\mathfrak{a}_i)) + d^2 C)$. As $\mathbf{S}(\beta \alpha_i) \leq \mathbf{S}(\beta) + \mathbf{S}(\mathfrak{a})/d + C$ the matrix B satisfies $\log(|B|) \leq \max(\mathbf{S}(\alpha_{i,j}))/d + \max(\mathbf{S}(\mathfrak{a}_i))/d^2 + C/d$. Since we know that the matrix B has determinant $\mathbf{N}(\mathfrak{d})$, where \mathfrak{d} denotes the determinantal ideal of $(A, (\mathfrak{a}_i)_i)$, computing the HNF over the integers has complexity in

$$\tilde{O}((dn)^2 \log(|B|) + (dn)^3 \log(\mathbf{N}(\mathfrak{d}))) \subseteq \tilde{O}(dn^2 \max \mathbf{S}(\alpha_{i,j}) + n^2 \max \mathbf{S}(\mathfrak{a}_i) + d^2 n^3 \mathbf{S}(\mathfrak{d}) + d^2 n C).$$

Combining this with the complexity of computing B we get an overall complexity in

$$\tilde{O}(d^2 n^2 \max \mathbf{S}(\alpha_{i,j}) + dn^2 \max \mathbf{S}(\mathfrak{a}_i) + d^2 n^3 \mathbf{S}(\mathfrak{d}) + d^3 n^2 C).$$

While the dependency on n is the same as in the pseudo-HNF case (see Theorem 34), the powers of d are slightly lower due to the absence of ideal arithmetic involving normalization and reduction. We conclude that one should always use the HNF over the rational integers if possible. Note that this discussion depends on the chosen pseudo-HNF algorithm and not on the notion of the pseudo-HNF itself and of course it is possible that more sophisticated approaches yield different conclusions.

7. The pseudo-SNF algorithm

The notion of pseudo-Smith normal form (pseudo-SNF) was introduced by Cohen [7], see also [8, Sec. 1.7], to describe quotients of \mathcal{O}_K -modules and to generalize the Smith normal form to modules over Dedekind domains. For simplicity, we restrict ourselves to quotients of modules of the same rank, but these results can easily be generalized to quotients of the form M/N where $\text{rank}(M) > \text{rank}(N)$. Let $A \in K^{n \times n}$ be a non-singular matrix and $I = (\mathfrak{b}_1, \dots, \mathfrak{b}_n)$, $J = (\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ families of fractional ideals. We say that (A, I, J) is an integral bi-pseudo matrix for the rank- n \mathcal{O}_K -modules

$$\begin{aligned} M &= \mathfrak{b}_1 \eta_1 \oplus \dots \oplus \mathfrak{b}_n \eta_n \\ N &= \mathfrak{a}_1 \omega_1 \oplus \dots \oplus \mathfrak{a}_n \omega_n \end{aligned}$$

if $a_{i,j} \in \mathfrak{b}_i \mathfrak{a}_j^{-1}$ for all $1 \leq i, j \leq n$, and the linear transformation $f : K^n \rightarrow K^n$ associated to A satisfies

$$f(\omega_j) = a_{1,j} \eta_1 + \dots + a_{n,j} \eta_n$$

for all $1 \leq j \leq n$. Then, it is shown in [8, Sec. 1.7] that the quotient Q associated to (A, I, J) is

$$Q = (\mathfrak{b}_1 \eta_1 \oplus \dots \oplus \mathfrak{b}_n \eta_n) / (\mathfrak{a}_1 f(\omega_1) \oplus \dots \oplus \mathfrak{a}_n f(\omega_n)).$$

In the case of two modules M, N of rank n with $N \subset M$, the elementary divisor theorem ensures the existence of $(\alpha_1, \dots, \alpha_n) \in K^n$, fractional ideals $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ and integral ideals $(\mathfrak{d}_1, \dots, \mathfrak{d}_n)$ with $\mathfrak{d}_{i-1} \subset \mathfrak{d}_i$ such that

$$M = \mathfrak{b}_1 \alpha_1 \oplus \dots \oplus \mathfrak{b}_n \alpha_n \text{ and } N = \mathfrak{d}_1 \mathfrak{b}_1 \alpha_1 \oplus \dots \oplus \mathfrak{d}_n \mathfrak{b}_n. \quad (7)$$

In the language of bi-pseudo matrices the elementary divisor theorem takes the following form.

Theorem 38 (Definition of the pseudo-SNF). *Let (A, I, J) be an integral bi-pseudo matrix with $I = (\mathfrak{b}_i)_{i \leq n}$ and $J = (\mathfrak{a}_i)_{i \leq n}$. There exist ideals $(\mathfrak{b}'_i)_{i \leq n}$, $(\mathfrak{a}'_i)_{i \leq n}$ and $n \times n$ matrices U, V such that with $\mathfrak{d}_i = \mathfrak{a}'_i \mathfrak{b}'_i{}^{-1}$ we have*

1. $\prod_i \mathfrak{a}_i = \det(U) \prod_i \mathfrak{a}'_i$ and $\prod_i \mathfrak{b}'_i = \det(V) \prod_i \mathfrak{b}_i$.
2. $V A U$ is the $n \times n$ identity matrix.
3. The \mathfrak{d}_i are integral and $\mathfrak{d}_{i-1} \subset \mathfrak{d}_i$ for $2 \leq i \leq n$.
4. For all i, j , $u_{i,j} \in \mathfrak{a}_i \mathfrak{a}'_j{}^{-1}$ and $v_{i,j} \in \mathfrak{b}'_i \mathfrak{b}_j{}^{-1}$.

In this case, the triplet $(U A V, (\mathfrak{b}'_i)_{i \leq n}, (\mathfrak{a}'_i)_{i \leq n})$ is called a pseudo-SNF of (A, I, J) .

Our algorithm for computing the pseudo-SNF of an integral bi-pseudo matrix is derived from [8, Alg. 1.7.3]. The possibility of working modulo the determinantal ideal was considered (although not explicitly described), but as for the computation of the pseudo-HNF, this does not prevent the growth of the denominators. We propose a modular version of [8, Alg. 1.7.3] incorporating the

Algorithm 6 pseudo-SNF of a full-rank square bi-pseudo matrix

Input: $(A, (\mathbf{b}_k)_{k \leq n}, (\mathbf{a}_k)_{k \leq n})$ integral $n \times n$ bi-pseudo matrix and $\mathfrak{d} = \det(A) \prod_i \mathbf{a}_i \mathbf{b}_i^{-1}$.

- 1: Compute $(\mathbf{b}_i^{-1})_{i \leq n}$.
- 2: Normalize (A'_j, \mathbf{a}_j) and (A_i, \mathbf{b}_i^{-1}) for $i, j \leq n$.
- 3: Reduce $a_{i,j} \bmod \mathbf{a}_i^{-1} \mathbf{b}_j$ for $i, j \leq n$.
- 4: **for** $i = n, \dots, 1$ **do**
- 5: StepOver \leftarrow false
- 6: **while** StepOver = false **do**
- 7: $M \leftarrow \text{ColPivot}(M, \mathfrak{d}, i)$ with Algorithm 7.
- 8: $M, \text{StepOver} \leftarrow \text{RowPivot}(M, \mathfrak{d}, i)$ with Algorithm 8.
- 9: **if** StepOver = true **then**
- 10: $\mathbf{b} \leftarrow \mathbf{a}_i \mathbf{b}_i$.
- 11: **for** $1 \leq k, l < i$, StepOver = true **do**
- 12: **if** $a_{k,l} \mathbf{a}_l \mathbf{b}_k^{-1} \notin \mathbf{b}$ **then**
- 13: Let $g \in \mathfrak{g} := \mathbf{b}_i \mathbf{b}_k^{-1}$ such that $a_{k,l} g \notin \mathbf{a}_i \mathbf{a}_l^{-1}$.
- 14: $A_i \leftarrow A_i + g A_k$, StepOver \leftarrow false.
- 15: Reduce $a_{k,i} \bmod \mathfrak{d} \mathbf{a}_k^{-1} \mathbf{b}_i$ for $k \leq i$ using Algorithm 1.
- 16: **end if**
- 17: **end for**
- 18: **end if**
- 19: **if** StepOver = true **then**
- 20: $\mathbf{a}_i \leftarrow a_{i,i} \mathbf{a}_i$, $a_{i,i} \leftarrow 1$.
- 21: $\mathfrak{d}_i \leftarrow \gcd(\mathbf{a}_i \mathbf{b}_i^{-1}, \mathfrak{d})$.
- 22: $\mathfrak{d} \leftarrow \mathfrak{d} \mathfrak{d}_i^{-1}$.
- 23: **end if**
- 24: **end while**
- 25: **end for**
- 26: **return** $\mathfrak{d}_1, \dots, \mathfrak{d}_n$.

normalization. We restrict ourselves to the case of non-singular square integral pseudo-matrices, but this result can be extended to the rectangular case easily.

For the sake of clarity, we give the description of row and column operations in separate algorithms. These are very similar to the row operations for the pseudo-HNF computation. The main difference is that we cannot use normalization on the pivots since we need a strictly increasing chain of ideals to ensure that the algorithm terminates.

Before proving the correctness of Algorithm 6, let us prove that the elementary operations performed on the bi-pseudo matrix do not change the quotient module that it represents.

Lemma 39. *Let $A, (\mathbf{b}_i)_{i \leq n}, (\mathbf{a}_i)_{i \leq n}$ be a bi-pseudo matrix. Then the operations*

- $\mathbf{b}_i \leftarrow (\alpha) \mathbf{b}_i$, $A_i \leftarrow (\alpha) A_i$,
- $\mathbf{a}_j \leftarrow (\alpha) \mathbf{a}_j$, $A'_j \leftarrow (1/\alpha) A'_j$,

Algorithm 7 ColPivot

Input: $(A, (\mathbf{b}_k)_{k \leq n}, (\mathbf{a}_k)_{k \leq n}), i \leq n, \mathfrak{d}$

- 1: **for** $j = i - 1, \dots, 1$ **do**
- 2: **if** $a_{i,j} \neq 0$ **then**
- 3: $\mathbf{g} \leftarrow a_{i,i} \mathbf{a}_i + a_{i,j} \mathbf{a}_j$
- 4: Compute $\gamma \in \mathbf{a}_i \mathbf{g}^{-1}$ and $\delta \in \mathbf{a}_j \mathbf{g}^{-1}$ such that $a_{i,i} \gamma + a_{i,j} \delta = 1$ using Algorithm 4.
- 5: $(A'_j, A'_i) \leftarrow (a_{i,j} A'_j - a'_{i,i} A_i, \gamma A'_i + \delta A'_j)$.
- 6: $(\mathbf{a}_j, \mathbf{a}_i) \leftarrow (\mathbf{a}_i \mathbf{a}_j \mathbf{g}^{-1}, \mathbf{g})$.
- 7: Normalize (A'_j, \mathbf{a}_j) and (A'_i, \mathbf{a}_i) using Algorithm 2.
- 8: Reduce $a_{j,k} \bmod \mathfrak{d} \mathbf{a}_j^{-1} \mathbf{b}_k$ and $a_{i,k} \bmod \mathfrak{d} \mathbf{a}_i^{-1} \mathbf{b}_k$ for $k \leq i$ using Algorithm 1.
- 9: **end if**
- 10: **end for**

do not modify the quotient module. In particular, the operations

- Normalize (A_i, \mathbf{b}_i^{-1}) using Algorithm 2.
- Normalize (A'_j, \mathbf{a}_j) using Algorithm 2.

do not modify the quotient module. Moreover, they leave the integral ideal $\sum_{i,j} a_{i,j} \mathbf{a}_j \mathbf{b}_i^{-1}$ unchanged.

Proof. We keep the same notation as before: $M = \bigoplus_i \mathbf{b}_i \eta_i$, $N = \bigoplus_i \mathbf{a}_i \omega_i$. Then if we perform $\mathbf{b}_i \leftarrow (\alpha) \mathbf{b}_i$, we need to update $\eta_i \leftarrow \frac{1}{\alpha} \eta_i$ to preserve M . Then for all $1 \leq j \leq n$,

$$\begin{aligned} f(\omega_j) &= a_{1,j} \eta_1 + \dots + a_{n,j} \eta_n \\ &= a_{1,j} \eta_1 + \dots + a_{i,j} \alpha \left(\frac{\eta_i}{\alpha} \right) + \dots + a_{n,j} \eta_n. \end{aligned}$$

Thus the i -th row of A gets multiplied by α .

Likewise, if $\mathbf{a}_j \leftarrow (\alpha) \mathbf{a}_j$, we need to update $\omega_j \leftarrow \frac{1}{\alpha} \omega_j$ to preserve N , which means by linearity that $f(\omega_j) = \frac{1}{\alpha} f(\omega_j)$. This means that the j -th column of A gets multiplied by $\frac{1}{\alpha}$. \square

Proposition 40. *Algorithm 6 terminates and gives the pseudo-SNF of the input.*

Proof. The proof of the termination of the non-modular version of Algorithm 6 is given in the proof of [8, Alg. 1.7.3], while the correctness of the modular version is treated in the integer case in the proof of [7, Alg. 2.4.14]. We only highlight the points where our context could induce a difference. The main argument showing that Steps 3 to 19 will only be repeated a finite amount of times is that the integral ideal $a_{i,i} \mathbf{a}_i \mathbf{b}_i^{-1}$ at step i either increases or is left unchanged therefore triggering the end of the loop. The main difference with [8, Alg. 1.7.3]

Algorithm 8 RowPivot

Input: $(A, (\mathbf{b}_k)_{k \leq n}, (\mathbf{a}_k)_{k \leq n}), i \leq n, \mathfrak{d}$

```

1: StepOver  $\leftarrow$  true
2: for  $j = i - 1, \dots, 1$  do
3:   if  $a_{j,i} \neq 0$  then
4:      $\mathbf{g} \leftarrow \mathbf{b}_i^{-1} + a_{j,i} \mathbf{b}_j^{-1}$ 
5:     Compute  $\gamma \in \mathbf{b}_i^{-1} \mathbf{g}^{-1}$  and  $\delta \in \mathbf{b}_j \mathbf{g}^{-1}$  such that  $\gamma + a_{j,i} \delta = 1$  using
       Algorithm 4.
6:      $(A_j, A_i) \leftarrow (a_{j,i} A_j - a_{i,i} A_i, \gamma A_i + \delta A_j)$ .
7:      $(\mathbf{b}_j, \mathbf{b}_i) \leftarrow (\mathbf{b}_i \mathbf{b}_j \mathbf{g}, \mathbf{g}^{-1})$ .
8:     Normalize  $(A_j, \mathbf{b}_j^{-1})$  and  $(A_i, \mathbf{b}_i^{-1})$  using Algorithm 2.
9:     Reduce  $a_{k,j} \bmod \mathfrak{d} \mathbf{a}_k^{-1} \mathbf{b}_j$  and  $a_{k,i} \bmod \mathfrak{d} \mathbf{a}_k^{-1} \mathbf{b}_i$  for  $k \leq i$  using Algo-
       rithm 1.
10:    StepOver  $\leftarrow$  false.
11:  end if
12: end for
13: return StepOver.

```

is that we normalize and reduce \mathbf{a}_j, A_j for $j \leq i$ to prevent coefficient explosion. Without taking into account modular reductions, Steps 5 and 6 transform the triplet $(a_{i,i}, \mathbf{a}_i, \mathbf{b}_i^{-1})$ into

$$\left(\frac{1}{\alpha_i \beta_i}, \sum_{j \leq i} a_{i,j} (\alpha_j) \mathbf{a}_j, \beta_i \mathbf{b}_i^{-1} + \sum_{j < i} a'_{j,i} (\beta_j) \mathbf{b}_j^{-1} \right),$$

where the $a'_{i,j}$ are the entries of A after Step 5, α_j are the minima used to normalize \mathbf{a}_j, A'_j and β_j are the minima used to normalize \mathbf{b}_i^{-1}, A_i . As in [8, Alg. 1.7.3], $a_{i,i} \mathbf{a}_i \mathbf{b}_i^{-1} \subseteq \sum_{i,j} a_{i,j} \mathbf{a}_j \mathbf{b}_i^{-1} \subseteq \mathcal{O}_K$ is integral since according to Lemma 39 the normalization steps do not alter this property. In addition, we see that $a_{i,i} \mathbf{a}_i \mathbf{b}_i^{-1}$ can only increase. To show that the modular reductions do not prevent the algorithm to terminate, we compare the evolution of $a_{i,i} \mathbf{a}_i \mathbf{b}_i^{-1}$ and $\overline{a_{i,i}} \mathbf{a}_i \mathbf{b}_i^{-1}$ where the $a_{i,j}$ are the coefficients of the matrix A during the course of Algorithm 6 executed without the modular reductions while the $\overline{a_{i,j}}$ are the same values when Algorithm 6 is run with modular reductions. The analysis of [8, Alg. 1.7.3] still holds for the non-modular version of Algorithm 6 which only differs from [8, Alg. 1.7.3] by the normalizations. The essential argument for the termination of Algorithm 6 is that we have

$$a_{i,i} \mathbf{a}_i \mathbf{b}_i^{-1} \subseteq \overline{a_{i,i}} \mathbf{a}_i \mathbf{b}_i^{-1} + \mathfrak{d} \subseteq \mathcal{O}_K.$$

Indeed, let $\overline{a_{i,j}} := a_{i,j} \bmod \mathfrak{d} \mathbf{a}_j^{-1} \mathbf{b}_i$, and $d_{i,j} := \overline{a_{i,j}} - a_{i,j} \in \mathfrak{d} \mathbf{a}_j^{-1} \mathbf{b}_i$, then for

each $x \in \mathfrak{a}_i \mathfrak{b}_i^{-1}$,

$$\begin{aligned} a_{i,i}x &= \underbrace{\overline{a_{i,j}}x}_{\in \overline{a_{i,j}} \mathfrak{a}_i \mathfrak{b}_i^{-1}} + \underbrace{d_{i,i}x}_{\in \mathfrak{d}} \in \overline{a_{i,i}} \mathfrak{a}_i \mathfrak{b}_i^{-1} + \mathfrak{d} \\ \overline{a_{i,i}}x &= \underbrace{a_{i,i}x}_{\in \mathcal{O}_K} - \underbrace{d_{i,i}x}_{\in \mathcal{O}_K} \in \mathcal{O}_K. \end{aligned}$$

Therefore, $\overline{a_{i,i}} \mathfrak{a}_i \mathfrak{b}_i^{-1} + \mathfrak{d}$ is an integral ideal which strictly increases and can only stabilize when the termination condition is reached.

The other main claim to be verified is the correctness of the modular approach. We adapt and reuse the argument presented in the proof of [7, Alg. 2.4.14]. We extend the notion of an $i \times i$ submatrix to pseudo-matrices by taking into account the coefficient ideals. Then let $\delta_i(A, I, J)$ be the sum of the determinantal ideal of all the $i \times i$ submatrices of (A, I, J) . As in the integer case, this value is an integral ideal invariant under the transformations performed in the non-modular version of Algorithm 6. In addition, we need to prove that the modular reductions of the form $a_{i,j} \leftarrow a_{i,j} \bmod \mathfrak{d} \mathfrak{a}_j^{-1} \mathfrak{b}_j$ for $\mathfrak{d} \subseteq \mathcal{O}_K$ performed on rows and columns preserve $\gcd(\det(A) \prod_i \mathfrak{a}_i \mathfrak{b}_i^{-1}, \mathfrak{d})$. From the symmetry between row and column reduction, it suffices to prove this for row reductions. Our determinantal ideal is given by

$$\det(A) \prod_i \mathfrak{a}_i \mathfrak{b}_i^{-1} = \left(\sum_{\sigma \in S_n} \prod_i a_{i, \sigma(i)} \right) \prod_i \mathfrak{a}_i \mathfrak{b}_i^{-1}.$$

Let $\overline{a_{i,j}} := a_{i,j} \bmod \mathfrak{d} \mathfrak{a}_i^{-1} \mathfrak{b}_j$ and $d_{i,j} \in \mathfrak{d} \mathfrak{a}_i^{-1} \mathfrak{b}_j$ such that $\overline{a_{i,j}} = d_{i,j} + a_{i,j}$. In particular, for $\sigma \in S_n$, we have

$$\begin{aligned} & a_{1, \sigma(1)} \cdots \overline{a_{i, \sigma(i)}} \cdots a_{n, \sigma(n)} \prod_j \mathfrak{a}_j \mathfrak{b}_j^{-1} \\ &= a_{1, \sigma(1)} \cdots (d_{i, \sigma(i)} + a_{i, \sigma(i)}) \cdots a_{n, \sigma(n)} \prod_j \mathfrak{a}_j \mathfrak{b}_{\sigma(j)}^{-1} \\ &\subseteq \prod_j a_{j, \sigma(j)} \mathfrak{a}_j \mathfrak{b}_{\sigma(j)}^{-1} + d_{i, \sigma(i)} \mathfrak{a}_i \mathfrak{b}_{\sigma(i)}^{-1} \underbrace{\prod_{j \neq i} a_{j, \sigma(j)} \mathfrak{a}_j \mathfrak{b}_{\sigma(j)}^{-1}}_{\in \mathcal{O}_K} \\ &\subseteq \prod_j a_{j, \sigma(j)} \mathfrak{a}_j \mathfrak{b}_{\sigma(j)}^{-1} + \mathfrak{d}. \end{aligned}$$

This means that $\det(\overline{A}) \prod_i \mathfrak{a}_i \mathfrak{b}_i^{-1} \subseteq \det(A) \prod_i \mathfrak{a}_i \mathfrak{b}_i^{-1} + \mathfrak{d}$ where $\overline{A} = (\overline{a_{i,j}})_{i,j \leq n}$. We show by using the same argument that $\det(A) \prod_i \mathfrak{a}_i \mathfrak{b}_i^{-1} \subseteq \det(\overline{A}) \prod_i \mathfrak{a}_i \mathfrak{b}_i^{-1} + \mathfrak{d}$, thus concluding that $\det(\overline{A}) \prod_i \mathfrak{a}_i \mathfrak{b}_i^{-1} + \mathfrak{d} = \det(A) \prod_i \mathfrak{a}_i \mathfrak{b}_i^{-1} + \mathfrak{d}$. Let the $(\mathfrak{d}_i)_{i \leq n}$ be the elementary divisors of the quotient module represented by $(A, (\mathfrak{b}_i)_{i \leq n}, (\mathfrak{a}_i)_{i \leq n})$, and $\mathfrak{d} := \prod_i \mathfrak{d}_i$. Let $S = (I_n, (\mathfrak{b}'_i)_{i \leq n}, (\mathfrak{a}'_i)_{i \leq n})$, be the bi-pseudo matrix resulting from the manipulation described in Algorithm 6 on the input $(A, (\mathfrak{b}_i)_{i \leq n}, (\mathfrak{a}_i)_{i \leq n})$,

and Γ the actual pseudo-SNF of M . Then, as in the proof of [7, Alg. 2.4.14], we have

$$\begin{aligned}\mathfrak{d}_i \cdots \mathfrak{d}_n &= \gcd(\mathfrak{d}, \delta_{n-i+1}(\Gamma)) \\ &= \gcd(\mathfrak{d}, \delta_{n-i+1}(A, (\mathfrak{b}_i)_{i \leq n}, (\mathfrak{a}_i)_{i \leq n})) \\ &= \gcd(\mathfrak{d}, \delta_{n-i+1}(S)) \\ &= \gcd(\mathfrak{d}, \mathfrak{a}'_i \mathfrak{b}'_i{}^{-1} \cdots \mathfrak{a}'_n \mathfrak{b}'_n{}^{-1}).\end{aligned}$$

After setting $\mathfrak{P}_i = \mathfrak{d}_{i+1} \cdots \mathfrak{d}_n$ we have

$$\gcd(\mathfrak{d}\mathfrak{P}_i^{-1}, (\mathfrak{a}'_{i+1} \mathfrak{b}'_{i+1}{}^{-1} \cdots \mathfrak{a}'_n \mathfrak{b}'_n{}^{-1}) \mathfrak{P}_i^{-1}) = \mathcal{O}_K,$$

and

$$\gcd(\mathfrak{d}\mathfrak{P}_i^{-1}, (\mathfrak{a}'_i \mathfrak{b}'_i{}^{-1} \cdots \mathfrak{a}'_n \mathfrak{b}'_n{}^{-1}) \mathfrak{P}_i^{-1}) = \mathfrak{d}_i.$$

Therefore, $\mathfrak{d}_i = \gcd(\mathfrak{d}\mathfrak{P}_i^{-1}, \mathfrak{a}'_i \mathfrak{b}'_i{}^{-1})$, which shows by induction the correctness of Algorithm 6. \square

To analyze the cost of the pseudo-SNF computation, we first consider the blocks ColPivot and RowPivot which resemble the row operations performed during the pseudo-HNF computation. In the following, we keep the notations B_{id} and B_e from the analysis of the pseudo-HNF algorithm.

Proposition 41. *The cost of Algorithm 8 and Algorithm 7 is in*

$$\tilde{O}(d^3(d+n)(S(\mathfrak{d}) + d^4 + d^2 \log(|\Delta_K|) + dC)).$$

Proof. This is derived almost entirely from Lemma 33 which was used in the analysis of the pseudo-HNF algorithm. The only notable difference, which does not impact the complexity, occurs in the modular reduction (Step 8 of Algorithm 7 and Step 9 of Algorithm 8). First, the reduction is no longer modulo $\mathfrak{d}\mathfrak{b}_i^{-1}$ but modulo $\mathfrak{d}\mathfrak{a}_j^{-1}\mathfrak{b}_k$. However, the size of these ideals remains in $\tilde{O}(S(\mathfrak{d}) + B_{\text{id}})$ thus not impacting the analysis. Also, the reduction of the entries of a row or a column is modulo a different ideal for each entry, thus preventing us from reusing the reduced basis. However, considering the bounds on the size of the elements in play, this does not change the complexity. \square

Proposition 42. *The cost of Steps 10 to 18 of Algorithm 6 is in*

$$\tilde{O}(nd^2(d+n)(S(\mathfrak{d}) + d^4 + d^2 \log(|\Delta_K|)) + nd^3C).$$

Proof. Computing \mathfrak{b} costs $\tilde{O}(d^2 B_{\text{id}} + d^3 \log(|\Delta_K|) + d^3 C)$. It requires inverting \mathfrak{b}_i^{-1} and multiplying it with \mathfrak{a}_i . This is done only once.

Calculating $\mathfrak{a}_l \mathfrak{b}_k^{-1}$ does not involve inversion (we have \mathfrak{b}_k^{-1}) and thus costs $\tilde{O}(d^2 B_{\text{id}} + d^3 C)$. Then, calculating $a_{k,l} \mathfrak{a}_l \mathfrak{b}_k^{-1}$ costs $\tilde{O}(d^3 B_e)$. Checking whether $a_{k,l} \mathfrak{a}_l \mathfrak{b}_k^{-1} \subset \mathfrak{b}$ can be done by calculating the pseudo-HNF of $(H_1^t | H_2^t)^t$ where H_1 is the \mathbb{Z} -basis matrix of \mathfrak{b} and H_2 is that of $a_{k,l} \mathfrak{a}_l \mathfrak{b}_k^{-1}$. If it is the same as H_1 ,

it means that $a_{k,l}\mathbf{a}_l\mathbf{b}_k \subset \mathbf{b}$. The entries of the matrix representing the \mathbb{Z} -basis of \mathbf{b} have their size in $\tilde{O}(B_{\text{id}}/d^2)$, while the size of the entries of the matrix of the \mathbb{Z} -basis of $a_{k,l}\mathbf{a}_l\mathbf{b}_k^{-1}$ are in $\tilde{O}(B_{\text{id}}/d^2 + B_e + C/d)$. Therefore computing the HNF of their concatenation costs $\tilde{O}(dB_{\text{id}} + d^3B_e + d^2C) \subseteq \tilde{O}(d^3B_e)$. So the search for k, l such that $a_{k,l}\mathbf{a}_l\mathbf{b}_k^{-1} \not\subset \mathbf{b}$ costs $\tilde{O}(n^2d^3B_e)$.

Once we have k, l , we find g by checking if $a_{k,l}\alpha_h \in \mathbf{a}_i\mathbf{a}_l^{-1}$ where the $(\alpha_h)_{h \leq d}$ are the elements of the \mathbb{Z} -basis of \mathfrak{g} . Calculating \mathfrak{g} and $\mathbf{a}_i\mathbf{a}_l^{-1}$ has the same cost as calculating \mathbf{b} , that is $\tilde{O}(d^2B_{\text{id}} + d^3\log(|\Delta_K|) + d^3C)$. As $S(a_{k,l}\alpha_h) \in \tilde{O}(B_e + B_{\text{id}}/d + C)$, the entries of the corresponding vector are in $\tilde{O}(B_e/d + B_{\text{id}}/d^2 + C/d)$. Likewise, the entries of the matrix of the \mathbb{Z} -basis of $\mathbf{a}_i\mathbf{a}_l^{-1}$ are in $\tilde{O}(B_{\text{id}}/d^2)$. Therefore, solving the linear system to verify if $a_{k,l}\alpha_h \in \mathbf{a}_i\mathbf{a}_l^{-1}$ costs $\tilde{O}(d^2B_e)$, and this is repeated at most d times, at a total cost of $\tilde{O}(d^3B_e)$. The resulting element $g \in \mathfrak{g}$ satisfies $S(g) \in \tilde{O}(B_{\text{id}}/d)$.

The step $A_i \leftarrow A_i + gA_k$ costs $\tilde{O}(d(d+n)B_{\text{id}}/d + dnB_e + d(n+d)C)$, and the resulting entries of $a_{k,i}$ of A_i satisfy $S(a_{k,i}) \in \tilde{O}(B_{\text{id}}/d + B_e + C) \subseteq \tilde{O}(B_e)$.

Finally, as $S(\mathfrak{d}\mathbf{a}_k^{-1}\mathbf{b}_i) \in \tilde{O}(S(\mathfrak{d}) + B_{\text{id}})$, the cost of the n reduction of $a_{k,i}$ modulo $\mathfrak{d}\mathbf{a}_k^{-1}\mathbf{b}_i$ is in

$$\tilde{O}(n(d^3(S(\mathfrak{d}) + B_{\text{id}}) + d^2B_e + d^3\log(|\Delta_K|) + d^3C)).$$

This and the term in $\tilde{O}(n^2d^3B_e)$ are the two dominant steps. The result follows by substituting the values of B_e and B_{id} by their expression in terms of the invariants of the field and $S(\mathfrak{d})$. \square

Proposition 43. *Let $B_A = \max_{i,j} S(a_{i,j})$ and $B_a = \max_{i,j} (S(\mathbf{a}_i), S(\mathbf{b}_j))$. The cost of Algorithm 6 is in*

$$\tilde{O}(nd((d+n)^2S(\mathfrak{d}) + nd^2)(d^4 + d^2\log(|\Delta_K|) + S(\mathfrak{d})) + n^2d^2(S(\mathfrak{d})C + B_A + B_a)).$$

Proof. First, let us estimate the cost of Steps 1 to 3. Inverting the \mathbf{b}_i costs $\tilde{O}(n(dB_a + d^3\log(|\Delta_K|) + d^2C))$. As in the proof of the complexity of Algorithm 5, the normalization costs $\tilde{O}(dn(d^2 + nB_a) + dn^2B_A + nd(d^2 + n)(\log(|\Delta_K|) + C))$.

The new elements have size $\tilde{O}(B_a + B_A + d\log(|\Delta_K|) + dC)$. Then, as we already have the \mathbf{b}_i^{-1} , calculating the $\mathfrak{d}\mathbf{a}_j\mathbf{b}_i^{-1}$ costs $\tilde{O}(n^2(d^2(B_{\text{id}} + S(\mathfrak{d})) + d^3C))$. Finally, the cost of the subsequent reduction is in

$$\tilde{O}(n^2d^2(d(d^4 + d^2\log(|\Delta|) + S(\mathfrak{d})) + B_A + B_a + dC)),$$

which is the dominant step of this precalculation.

Now, let us analyze the main loop of the algorithm. The condition `StepOver = true` is attained in at most $\tilde{O}(S(a_{i,i}\mathbf{a}_i\mathbf{b}_i^{-1}))$ since $a_{i,i}\mathbf{a}_i\mathbf{b}_i^{-1}$ becomes strictly larger at each iteration. Therefore, the number of iterations is in $\tilde{O}(\log(N(\mathfrak{d})))$. So Algorithm 8, Algorithm 7 and the Steps 10 to 18 are executed $\tilde{O}(nS(\mathfrak{d})/d)$ times. We obtain the cost of the main loop by adding the estimated costs found in Proposition 41 and Proposition 42 and multiplying this by $nS(\mathfrak{d})/d$. \square

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